

# Reductive group actions

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ABSTRACT. In this paper, we study rationality properties of reductive group actions which are defined over an arbitrary field of characteristic zero. Thereby, we unify Luna's theory of spherical systems and Borel-Tits' theory of reductive groups. In particular, we define for any reductive group action a generalized Tits index whose main constituents are a root system and a generalization of the anisotropic kernel. The index controls to a large extent the behavior at infinity (i.e., embeddings). For  $k$ -spherical varieties (i.e., where a minimal parabolic has an open orbit) we obtain explicit (wonderful) completions of the set of rational points. For local fields this means honest compactifications generalizing the maximal Satake compactification of a symmetric space. Our main tool is a  $k$ -version of the local structure theorem.

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## 1. Introduction

Let  $G$  be a connected reductive group. The main focus of this paper is to study the fine large scale geometry of homogeneous  $G$ -varieties when the ground field  $k$  is an arbitrary field of characteristic zero. When  $k$  is algebraically closed the last decades have seen tremendous progress in this direction. Important milestones were the theory of toroidal embeddings in [KKMSD73] (still serving as the blueprint for all later developments), the wonderful embeddings of symmetric varieties in [DCP83], and the groundbreaking work [LV83] of Luna and Vust describing all spherical embeddings and much more. Later Brion [Bri90] discovered that the asymptotic behavior of spherical varieties is determined by a

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system of spherical roots. This was subsequently generalized to arbitrary  $G$ -varieties in [Kno94].

Over non-closed fields much less is known even though the work of Satake [Sat60] on compactifications of symmetric spaces (where  $k = \mathbb{R}$ ) considerably predates the Luna-Vust theory. Here, mainly rationality questions are in the foreground, like which embeddings are defined over  $k$ , or which orbits carry  $k$ -rational points. One of the main obstacles is the fact that the theory over closed fields hinges crucially on the existence of a Borel subgroup. For that reason, the Luna-Vust theory extends quite easily to the case when  $G$  is split over  $k$ . This has enabled Sakellaridis and Venkatesh [SV12] to prove a Plancherel theorem for spherical varieties of split groups over  $p$ -adic fields. But even for quasi-split groups things become more difficult and for all other groups the theory has to be built up from scratch. The idea is of course to replace the Borel subgroup by a minimal parabolic subgroup  $P$ .

A first breakthrough in this direction was achieved over  $k = \mathbb{R}$  with the paper [KKS15a] where an  $\mathbb{R}$ -version of the local structure theorem of [BLV86] was proved and used to study the geometry of  $\mathbb{R}$ -spherical varieties. Later, it was successfully used in [KKSS15, KKSS14, KKS15b] for purposes of harmonic analysis.

In the present paper we go far beyond [KKS15a]. First, we replace  $\mathbb{R}$  by an arbitrary ground field of characteristic 0. Secondly, we have much more precise results on the structure of spherical embeddings and thirdly, most of our results are also valid in the non-spherical setting. Thereby we end up with a theory which unifies Luna’s theory of spherical varieties with the Borel-Satake-Tits theory [Tit59, Sat63, BT65, Tit66] of reductive groups over non-closed fields.

Since our approach deals mostly with rationality properties we will always assume that the  $G$ -variety  $X$  is  $k$ -dense, i.e., that  $X(k)$  is Zariski dense in  $X$ . Let  $P = MAN$  be a minimal parabolic  $k$ -subgroup of  $G$ . Here  $N$  is the unipotent radical,  $A$  is a maximal split torus and  $M$  is anisotropic. Then we construct a quite specific  $P$ -stable open subset  $X'$  of  $X$  such that the quotients  $X_{\text{el}} := X'/N$  and  $X_{\text{an}} := X'/AN$  exist, called the *elementary kernel* (an  $MA$ -variety) and the *anisotropic kernel of  $X$*  (an  $M$ -variety), respectively. The latter plays, like for reductive groups, the rôle of a black box about which we have little to say.

The canonical morphism  $X_{\text{el}} \rightarrow X_{\text{an}}$  is a principal bundle for a quotient  $A_k(X)$  of  $A$ . This torus  $A_k(X)$  is the analogue of  $A$  in  $G$ . Its character group  $\Xi_k(X) := \Xi(A_k(X))$  replaces the weight lattice of a group. Accordingly, the  $k$ -rank of  $X$  is defined as

$$(1.1) \quad \text{rk}_k X := \text{rk } \Xi_k(X) = \dim A_k(X) = \dim X_{\text{el}} - \dim X_{\text{an}}.$$

The root system is much more intricate to construct. For this, we start by considering a special kind of  $G$ -invariant valuations on the function field  $k(X)$  which we call  $k$ -central. By definition these are invariant valuations which are trivial on the subfield of  $AN$ -invariants. Each such valuation is related to a  $G$ -stable irreducible divisor  $D$  in some birational model of  $X$ . An important point is that for  $k$ -central valuations  $D$  is  $k$ -dense. This provides us with a tool to study rational points “at infinity”.

It is shown that  $k$ -central valuations are classified by elements of a certain subset  $\mathcal{Z}_k(X)$  of the vector space  $\mathcal{N}_k(X) := \text{Hom}(\Xi_k(X), \mathbb{Q})$ . A central fact (pun intended) is that  $\mathcal{Z}_k(X)$  is a Weyl chamber for a finite reflection group  $W_k(X)$  acting on  $\Xi_k(X)$ . Suitably

normalized, this defines a set  $\Sigma_k(X)$  of simple roots for a root system  $\Phi_k(X) \subset \Xi_k(X)$  with Weyl group  $W_k(X)$ . The elements of  $\Sigma_k(X)$  are called the *k-spherical roots of X*.

Now consider the algebraic closure  $K$  of  $k$ . Since everything can be done over  $K$ , as well there is also a root system  $\Phi_K(X)$  of  $X$ . Of course, this is just the one constructed previously by Brion. Another main result of this paper is that  $\Phi_k(X)$  is more or less the restricted root system of  $\Phi_K(X)$  to  $A$ . This was kind of unexpected since  $\Phi_K(X)$  is constructed using the action of minimal  $K$ -parabolic, i.e., a Borel subgroup on  $X$ . So there is no direct link between both root systems.

At this point, we have now four root systems to deal with, namely the two root systems of  $G$  (over  $K$  and  $k$ ) and the two root systems of  $X$ . They are subject to very strong compatibility constraints of which we prove a few. An exhaustive list will be subject of future research. It should be mentioned that the pair  $(\Phi_K(X), \Phi_k(X))$  looks very much like a Tits index of a reductive group but is in fact slightly more general.

We now come to the main goal of this paper namely analyzing the behavior of  $X$  at infinity. For this we construct for every fan  $\mathcal{F}$  which is supported in  $\mathcal{Z}_k(X)$  a normal partial compactification  $X(\mathcal{F})$  of  $X$ . This embedding comes with a stratification with one stratum  $X(\mathcal{C})$  for every cone  $\mathcal{C} \in \mathcal{F}$ . The main point is that every stratum is  $k$ -dense. Now assume that the support of  $\mathcal{F}$  is all of the valuation cone  $\mathcal{Z}_k(X)$ . Then it is *not true* that  $X(\mathcal{F})$  is a complete variety. Instead it is kind of complete with respect to rational points. This means that every normal completion of  $X(\mathcal{F})$  adds only very few rational points.

It is easiest to make this precise when  $X$  is *k-spherical*. This means by definition that  $P$  has an open orbit in  $X$ . Equivalently, the action of  $M$  on the anisotropic kernel of  $X$  is transitive. So, when  $X$  is  $k$ -spherical and  $\text{supp } \mathcal{F} = \mathcal{Z}_k(X)$  then we show that  $\overline{X}(k) = X(\mathcal{F})(k)$  for any normal (equivariant) completion  $\overline{X}$  of  $X(\mathcal{F})$ . Another consequence of  $k$ -sphericity is that the strata are precisely the  $G$ -orbits. In particular there are only finitely many of them. In contrast, it may happen that a completion has infinitely many orbits but the point is that almost none of them carries a  $k$ -rational point. Therefore we regard it as one of the main insights of this paper (and its precursor [KKS15a]) that in order to study rational points it is not necessary to construct true completions.

Now the main consequence of  $\mathcal{Z}_k(X)$  being a Weyl chamber is that it is cosimplicial, i.e., defined by a set of linearly independent inequalities. A particularly favorable situation arises if  $\mathcal{Z}_k(X)$  is strictly convex. In this case  $X$  will be called *k-convex*. This condition is rather mild, since every  $G$ -variety can be made  $k$ -convex by dividing by a suitable split torus. In this case  $\mathcal{Z}_k(X)$  is also simplicial and one can take as a fan the set of all faces of  $\mathcal{Z}_k(X)$ . We call this the *standard fan  $\mathcal{F}_{\text{st}}$  of  $X$*  and  $X_{\text{st}} := X(\mathcal{F}_{\text{st}})$  the *standard embedding*. Now for a  $k$ -convex,  $k$ -spherical homogeneous variety  $X$  its standard embedding has the following nice properties:

- The boundary  $X_{\text{st}} \setminus X$  consists of  $r = \text{rk}_k X$  irreducible components  $X_{\{1\}}, \dots, X_{\{r\}}$ .
- The orbit closures are precisely the set-theoretic intersections  $X_I := \bigcap_{i \in I} X_{\{i\}}$  where  $I$  runs through all subsets of  $\{1, \dots, r\}$ .
- Each  $X_I$  is  $k$ -dense and of codimension  $|I|$  in  $X_{\text{st}}$ .
- Let  $X_{\text{st}} \subseteq \overline{X}$  be an equivariant embedding. Then  $\overline{X}(k) = X_{\text{st}}(k)$ .

Often the standard embedding is even smooth. In this case, we show that all orbit closures  $X_I$  are smooth and that the intersections above are even scheme-theoretic. Following

DeConcini and Procesi, [DCP83], the varieties  $X$  and  $X_{\text{st}}$  will then be called  $k$ -wonderful. Even that condition is quite mild: we show that for every  $k$ -convex variety  $X$  there is a finite abelian group  $E$  of automorphisms such that  $X/E$  is wonderful.

The paper contains a couple more topics which have not yet been mentioned. In Section 8 we classify two types of varieties. The first are spaces whose  $k$ -rank is zero. We show that they are all parabolically induced from anisotropic actions. Their importance lies in the fact that the closed stratum of a standard embedding is of rank zero. The second, more general type are the so-called  $k$ -horospherical varieties. They are characterized by  $\mathcal{Z}_k(X)$  being the entire space  $\mathcal{N}_k(X)$  or, equivalently, that the root system is empty. We show that all  $k$ -horospherical varieties are parabolic inductions of elementary actions. Geometrically they appear as maximal degenerations of spherical varieties. In Section 12 we consider more general degenerations. More precisely, we construct a smooth morphism  $\mathfrak{X} \rightarrow \mathfrak{Y}$  with projective base whose fibers are called the *boundary degenerations of  $X$* .

In Section 13 we generalize the main result of [KKSS15] from  $\mathbb{R}$  to local fields of characteristic zero. More precisely, we prove the a weak version of the polar decomposition. Finally, we show how to obtain the classical maximal Satake compactification of a Riemannian symmetric space with our approach. Moreover, we compare the standard embedding of a variety  $X$  considered as a variety over  $k$  with the standard embedding over the algebraic closure. They are related by a birational morphism from the former to the later which is in general not an open embedding. We also give an examples in which the standard embedding over  $K$  has bad rationality properties.

Some words to the proofs: As already mentioned everything hinges on the  $k$ -Local Structure Theorem. Here we follow the proof of [KKS15a] which is itself an adaption of the proof in [Kno94]. The main difficulty is to show the existence of sufficiently many  $P$ -eigenfunctions on  $X$ . For this we replace the integration and positivity argument in [KKS15a] by an argument using Kempf’s theory of optimal 1-parameter subgroups [Kem78].

Secondly, even though the main applications are for  $k$ -spherical varieties the theory does not simplify very much by sticking to the spherical case. The reason for this is that  $k$ -spherical varieties do not stay spherical when passing to the algebraic closure. So we do need to a considerable extent the embedding theory of non-spherical varieties.

Finally, the order in which the theory evolves is different than the one in the exposition above where we first described the root system and then derived properties of embeddings. In practice we prove a fair amount of properties of the compactification first and then reinterpret them in terms of the root system.

## 2. Notation

In the entire paper,  $k$  is a field of characteristic zero. Its algebraic closure is denoted by  $K$  and its Galois group by  $\mathcal{G} := \text{Gal}(K|k)$ .

Our point of view will be that a  $k$ -variety is variety over  $K$  equipped with a compatible  $\mathcal{G}$ -action. Accordingly, for us the terms “connected” and “irreducible” mean “absolutely connected” and “absolutely irreducible”. All  $k$ -varieties are, by definition, non-empty and irreducible.

Let  $H$  be a linear algebraic group defined over  $K$ . Then  $\Xi(H) := \text{Hom}(H, \mathbf{G}_m)$  denotes its character group. If  $H$  is defined over  $k$ , then  $\Xi(H)$  carries a  $\mathcal{G}$ -action. The space  $\mathcal{N}(H) := \text{Hom}(\Xi(H), \mathbb{Q})$  denotes the space of rational cocharacters of  $H$ .

Let  $V$  be a representation of  $H$  and  $\chi \in \Xi(H)$ . Then  $V_\chi^{(H)}$  denotes the set of  $\chi$ -eigenvectors of  $V$ . Observe that  $V_\chi^{(H)}$  never contains the vector 0. Thus, for the trivial character  $\chi = 1$  we have  $V_1^{(H)} = V^H \setminus \{0\}$ . We denote by  $V^{(H)}$  the union of all  $V_\chi^{(H)}$  with  $\chi \in \Xi(H)$ . For  $v \in V^{(H)}$  let  $\chi_v$  be the unique character with  $v \in V_{\chi_v}^{(H)}$ .

In the whole paper,  $G$  is a connected reductive  $k$ -group. We fix a minimal parabolic  $k$ -subgroup  $P \subseteq G$  with decomposition  $P = MAN$  where  $N = R_u P$  is the unipotent radical,  $A$  is a maximal split subtorus and  $M$  is anisotropic. Hence  $MA$  is a Levi complement of  $P$ . Let  $T \subseteq MA$  be a maximal  $k$ -subtorus. Then  $A \subseteq T$  and restriction to  $A$  defines a surjective homomorphism

$$(2.1) \quad \text{res}_A : \Xi(T) \rightarrow \Xi(A).$$

If  $\Omega \subseteq \Xi(T)$  is any subset we adopt the notation

$$(2.2) \quad \text{res}'_A \Omega := (\text{res}_A \Omega) \setminus \{0\}.$$

Let  $\Phi = \Phi(G, T)$  be the associated root system and

$$(2.3) \quad \Phi_k := \text{res}'_A \Phi$$

the restricted root system (in general not reduced). The Weyl group of  $\Phi$  is denoted by  $W = W(G)$ . Choosing a Borel subgroup  $B$  with  $T \subseteq B \subseteq P$  yields a set of simple roots  $S \subseteq \Phi$  and a set of simple restricted roots

$$(2.4) \quad S_k := \text{res}'_A S \subseteq \Phi_k.$$

On the other side, the elements of

$$(2.5) \quad S^0 := \{\alpha \in S \mid \text{res}_A \alpha = 0\}$$

are the compact simple roots which are the simple roots of  $M$ . Reflections about elements of  $S^0$  generate the Weyl group  $W^0 \subseteq W$  of  $M$ .

The Galois group  $\mathcal{G}$  acts on  $\Xi(T)$  leaving the root system  $\Phi$  invariant. The restriction map  $\text{res}_A$  is  $\mathcal{G}$ -invariant. There is also the so-called  $*$ -action of  $\mathcal{G}$  on  $\Xi(T)$  which is constructed as follows: for each  $\gamma \in \mathcal{G}$  there is a unique element  $w_\gamma \in W$  with  $w_\gamma \gamma S = S$ . Then one defines

$$(2.6) \quad \gamma * \chi := w_\gamma \gamma(\chi).$$

In the following, we refer to  $\mathcal{G}$  acting by the  $*$ -action as  $\mathcal{G}^*$ .

It is known that  $w_\gamma \in W^0$  for all  $\gamma \in \mathcal{G}$  ([BT65, Proof of Prop. 6.7] or [Spr98, Proof of Prop. 15.5.3]). This means that if

$$(2.7) \quad \Delta^* := \{\varphi \in \text{Aut } \Xi(T) \mid \varphi(S) = S, \varphi(S^0) = S^0\}$$

is the group of ‘‘graph automorphisms’’ then the  $\mathcal{G}^*$ -action is a homomorphism  $\mathcal{G} \rightarrow \Delta^*$  while the  $\mathcal{G}$ -action is a lift to  $\Delta^* \ltimes W^0$ . As a consequence the ordinary and the  $*$ -action coincide on  $\Xi(T)^{W^0} = \Xi(T/(T \cap M^0)^0)$ .

Throughout this paper we denote algebraic groups by upper case Latin letters and their corresponding Lie algebras by lower case German letters, e.g.  $\mathfrak{h}$  is the Lie algebra of  $H$ . The unipotent radical of an algebraic group  $H$  is denoted by  $R_u H$  or  $H_u$ .

### 3. Invariants of quasi-elementary groups

Following Borel and Tits, [BT65, Def. 4.23], we define

**3.1. Definition.** A linear algebraic  $k$ -group  $G$  is *anisotropic* if it is connected, reductive, and its  $k$ -rank is 0, i.e., it does not admit a non-trivial split subtorus.

According to [BT65, Cor. 8.5] an equivalent condition for a group  $G$  to be anisotropic is that all elements in  $G(k)$  are semisimple and that  $\text{Hom}_k(G, \mathbf{G}_m) = 1$ . Some authors, like, e.g., Springer in [Spr98, p. 271], drop the second condition. The ensuing notion is different but also very important for our purposes:

**3.2. Definition.** A linear algebraic  $k$ -group  $G$  is called *elementary* if it is connected and all of its  $k$ -rational elements are semisimple.

Clearly, elementary groups are reductive. The difference between an anisotropic and an elementary group is that the latter may have a non-trivial central split subtorus. Because of this reason, elementary groups are often called “anisotropic modulo center” (with reductivity tacitly being assumed). As an example, the group  $G = GL(1, \mathbb{H})$  is an elementary but non anisotropic  $\mathbb{R}$ -group. Over  $K$ , the algebraic closure of  $k$ , only the trivial group is anisotropic while exactly the tori are elementary.

These definitions extend to non-reductive groups as follows:

**3.3. Definition.** A  $k$ -group  $H$  with unipotent radical  $H_u$  is *quasi-anisotropic* or *quasi-elementary* if  $H/H_u$  is anisotropic or elementary, respectively.

Clearly, these two concepts imply connectivity. Since any parabolic subgroup of  $H$  contains  $H_u$ , it follows from [BT65, Cor. 4.17] that  $H$  is quasi-elementary if and only if it does not contain a proper parabolic  $k$ -subgroup. In particular, a minimal parabolic  $k$ -subgroup of any group is quasi-elementary and its Levi component is elementary.

The most general of these four concepts is that of a quasi-elementary groups. They have a factorization

$$(3.1) \quad H = MAN$$

where  $M$  is anisotropic,  $A$  is a split torus and  $N$  is the unipotent radical. This factorization is unique up to conjugation by  $N(k)$ . The group  $H$  is quasi-anisotropic, elementary, or anisotropic if and only if  $A$  is trivial,  $N$  is trivial, or both  $A$  and  $N$  are trivial, respectively.

It follows from the definition that  $k$ -subgroups of elementary groups are elementary. In particular, they are reductive, as well.

For any connected group  $H$  let  $H_{\text{el}} \subseteq H$  be the Zariski closure of the subgroup generated by all  $k$ -rational unipotent elements. Then  $H_{\text{el}}$  is the smallest normal subgroup of  $H$  such that  $H/H_{\text{el}}$  is elementary. We call it the *elementary radical of  $H$* . Concretely, the elementary radical can also be described as follows: if  $H = LH_u$  is the Levi decomposition of  $H$  then  $H_{\text{el}} = L_n H_u$  where  $L_n \subseteq L$  is the product of all non-anisotropic simple factors of  $L$ .

This has the following consequence: Let  $P = MAN \subseteq H$  be a minimal parabolic  $k$ -subgroup. Since  $MA$  contains all anisotropic simple factors and also the center of  $L$  we

get

$$(3.2) \quad H = MA \cdot H_{\text{el}},$$

There is an analogous notion for anisotropic groups: every connected group  $H$  has a smallest normal subgroup  $H_{\text{an}}$  such that  $H/H_{\text{an}}$  is anisotropic. We call  $H_{\text{an}}$  the *anisotropic radical of  $H$* . As a group it is generated by  $H_u$  and all split subtori. If  $P = MAN$  is a minimal parabolic subgroup then

$$(3.3) \quad H_{\text{an}} = A \cdot H_{\text{el}} \text{ and } H = M \cdot H_{\text{an}}.$$

The characterization of quasi-elementary groups in terms of parabolic subgroups can be extended as follows:

**3.4. Proposition.** *For a connected  $k$ -group  $H$ , the following are equivalent:*

- i)  $H$  is quasi-elementary.*
- ii) Let  $F \subseteq H$  be any  $k$ -subgroup. Then  $H/F$  is an affine variety.*
- iii)  $H$  does not contain any proper parabolic  $k$ -subgroup.*

*Proof.* Assume that  $H = MAN$  is quasi-elementary and let  $F \subseteq H$  be a  $k$ -subgroup. Consider the morphism

$$(3.4) \quad \pi : H/F \rightarrow H/FN \cong (H/N)/(FN/N).$$

The group  $FN/N$  is a subgroup of the elementary group  $H/N = MA$  and therefore reductive. This implies that  $H/FN$  is an affine variety. The fiber of  $\pi$  is  $FN/F \cong N/F \cap N$ . It is the orbit of a unipotent group, hence affine, as well. This implies that  $\pi$  is an affine morphism on an affine variety. Hence  $H/F$  is affine.

Now assume *ii)*. Let  $P \subseteq H$  be a minimal parabolic  $k$ -subgroup. Then *ii)* applied to  $F = P$  implies that  $H/P$  is both affine and complete and therefore a point. Thus  $P = H$ .

Finally, the equivalence of *i)* and *iii)* follows from [BT65, Cor. 4.17], as already mentioned above.  $\square$

**3.5. Definition.** A  $k$ -variety  $X$  is called  *$k$ -dense* if the set  $X(k)$  of  $k$ -rational points is Zariski dense in  $X$ .

Clearly, the affine space and the projective space is  $k$ -dense. If  $X \rightarrow Y$  is a dominant  $k$ -morphism and  $X$  is  $k$ -dense then  $Y$  is  $k$ -dense, as well. The product  $X \times Y$  of two  $k$ -dense varieties is  $k$ -dense. More generally, if  $X \rightarrow Y$  is a Zariski trivial fiber bundle with fiber  $Z$  then  $X$  is  $k$ -dense if and only if both  $Y$  and  $Z$  are  $k$ -dense. This holds in particular for the total space of a line bundle over a  $k$ -dense variety.

In the remainder of this section we will study actions of quasi-elementary groups on (mostly) affine  $k$ -dense varieties. The point of departure will be the following fundamental theorem of Kempf:

**3.6. Theorem** ([Kem78, Cor. 4.3]). *Let  $G$  be a connected reductive  $k$ -group acting on an affine  $k$ -variety  $X$ . Let  $x \in X(k)$  be a  $k$ -rational point and let  $Y \subseteq X$  be a closed  $G$ -invariant  $k$ -subset with  $\overline{Gx} \cap Y \neq \emptyset$ . Then there exists a  $k$ -homomorphism  $\lambda : \mathbf{G}_m \rightarrow G$  such that  $\lim_{t \rightarrow 0} \lambda(t)x$  exists and lies in  $Y(k)$ .*

This implies:

**3.7. Corollary** ([Kem78, Remark after Cor. 4.4]). *Let  $G$  be an anisotropic group acting on a quasi-affine  $k$ -variety  $X$ . Then the  $G$ -orbit of any  $k$ -rational point  $x \in X(k)$  is closed in  $X$ .*

*Proof.* Since  $X$  is quasi-affine, it is isomorphic to a locally closed subvariety of an affine  $G$ -variety  $\overline{X}$ . Replacing  $X$  by  $\overline{X}$ , we may assume that  $X$  is affine. Now apply Kempf's theorem to  $Y := \overline{Gx} \setminus Gx$  and observe that any homomorphism  $\lambda : \mathbf{G}_m \rightarrow G$  is trivial since  $G$  is anisotropic. We conclude that  $Y$  is empty, i.e.,  $Gx$  is closed.  $\square$

Rosenlicht has proved in [Ros61] that also orbits of unipotent groups in quasi-affine varieties are closed. The following can be regarded as a combination of both cases:

**3.8. Theorem.** *Let  $H = MAN$  be a quasi-elementary group acting on a  $k$ -dense quasi-affine variety  $X$ . Then there is an affine  $H$ -variety  $Y$  with  $MN$  acting trivially, an  $H$ -equivariant morphism  $\pi : X \rightarrow Y$ , and an  $H$ -stable open dense subset  $Y_0 \subseteq Y$  such that for all  $y \in Y_0$  the fiber  $X_y$  is an  $MN$ -orbit.*

*Proof.* Choose an open equivariant embedding of  $X$  into an affine  $H$ -variety  $\overline{X}$ . Since it suffices to prove the assertion for  $\overline{X}$  we may assume from the outset that  $X$  is affine.

Now choose generators  $r_1, \dots, r_n$  of  $k(X)^N$ , the field of  $N$ -invariant rational functions. Since  $N$  is unipotent and since the ideal

$$(3.5) \quad \{f \in k[X] \mid fr_1, \dots, fr_n \in k[X]\}$$

is non-zero, it contains a non-zero  $N$ -invariant function  $f_0$ . By replacing  $r_1, \dots, r_n$  with  $f_0, f_0r_1, \dots, f_0r_n$  we may assume that all the functions  $r_i$  are in  $k[X]^N$ .

Let  $V \subseteq k[X]^N$  be the (finite dimensional)  $H$ -submodule of  $k[X]^N$  which is generated by  $\{r_1, \dots, r_n\}$ , let  $\mathcal{A} \subseteq k[X]^N$  be the (finitely generated) subalgebra which is generated by  $V$ , and let  $Z := \text{Spec } \mathcal{A}$  be the variety associated to  $\mathcal{A}$ . Then  $Z$  is an affine  $H$ -variety with  $N$  acting trivially. Moreover, the inclusion  $\mathcal{A} \hookrightarrow k[X]$  induces a dominant  $H$ -morphism  $\pi_N : X \rightarrow Z$ . By construction, this morphism induces an isomorphism of fields  $\pi_N^* : k(Z) \xrightarrow{\sim} k(X)^N$ . Rosenlicht's theorem [Ros56, Thm. 2] implies that the generic  $N$ -orbits are separated by rational  $N$ -invariants. This means that the generic fibers of  $\pi_N$  contain open dense  $N$ -orbits. Since orbits of a unipotent group on an affine variety are closed (by another theorem of Rosenlicht [Ros61], see also [Bor91, Prop. 4.10]), we see that the generic fibers of  $\pi_N$  are actually  $N$ -orbits.

Now consider the  $M$ -action on  $Z$  and let  $\pi_M : Z \rightarrow Y := Z//M$  be the categorical quotient. Observe that  $Y$  carries an action of  $H$  with  $MN$  acting trivially. All fibers of  $\pi_M$  contain a unique closed  $M$ -orbit. It follows from Corollary 3.7 that the orbits  $Mz$  with  $z \in Z(k)$  are closed. The  $k$ -density of  $X$  implies that  $Z$  is  $k$ -dense, as well. This implies that the generic  $M$ -orbits in  $Z$  are closed. Thus the generic fibers of  $\pi_M$  are precisely the generic  $M$ -orbits. Now we take for  $\pi$  the composition  $\pi_M \circ \pi_N : X \rightarrow Y$ .  $\square$

From this, we obtain the following generalization of Corollary 3.7:

**3.9. Corollary.** *Let  $H$  be a quasi-anisotropic group acting on a quasi-affine  $k$ -variety  $X$ . Then:*

- i) The orbit  $Hx$  of any  $x \in X(k)$  is closed in  $X$ .*



ii) Assume, moreover, that  $X$  is  $k$ -dense. Then the generic  $H$ -orbits are closed in  $X$ , i.e., there is an  $H$ -stable open dense subset  $X_0 \subseteq X$  such that all  $H$ -orbits of  $X_0$  are closed in  $X$ .

*Proof.* i) It is well known that  $H(k)$  is dense in  $H$  ([Bor91, Thm. 18.3]). Therefore, the orbit closure  $\overline{Hx}$  is  $k$ -dense. When one applies the theorem to  $\overline{Hx}$ , the variety  $Y$  is necessarily a point. Thus  $\overline{Hx}$  is an  $H$ -orbit, i.e.,  $Hx$  is closed.

ii) Since fibers are closed, this follows directly from the theorem.  $\square$

Next we establish a criterion for the existence of semiinvariants.

**3.10. Proposition.** *Let  $H$  be a quasi-elementary group acting on a  $k$ -dense affine variety  $X$ . Let  $I \subseteq K[X]$  be a non-zero  $H$ -stable ideal (not necessarily defined over  $k$ ). Then  $I$  contains an  $H$ -semiinvariant function (defined over  $k$ ).*

*Proof.* Let  $H = MAN$  be the factorization from (3.1) and  $\pi : X \rightarrow Y$  be the morphism as in Theorem 3.8. The zero locus  $Z$  of  $I$  in  $X$  is an  $H$ -stable proper closed subset. Its image  $\varphi(Z)$  in  $Y$  cannot be dense since otherwise  $Z$  would contain the generic  $MN$ -orbits and would therefore coincide with  $X$ . The ideal of all functions on  $Y$  which vanish in  $\pi(Z)$  is non-zero and therefore contains an  $A$ -semiinvariant function  $f_0$ . Then  $f_0$  is also  $H$ -semiinvariant since  $MN$  is acting trivially on  $Y$ . Now replace  $f_0$  by the (finite) product of all functions of the form  ${}^\gamma f_0$  where  $\gamma$  runs through the Galois group  $\text{Gal}(K|k)$ . This makes  $f_0$  being defined over  $k$ , as well. The pull-back  $f_1$  of  $f_0$  to  $X$  vanishes in  $Z$ . Hilbert's Nullstellensatz asserts that then a power  $f = f_1^N$  is in  $I$ .  $\square$

**3.11. Corollary.** *Let  $H = MAN$  be a quasi-elementary group acting on a  $k$ -dense affine variety  $X$ . Then either  $MN$  acts transitively on  $X$  or  $X$  carries a non-constant  $H$ -semiinvariant function defined over  $k$ .*

*Proof.* If  $MN$  acts transitively, then clearly all  $H$ -semiinvariants, being  $MN$ -invariant, are constant. Assume conversely that  $MN$  does not act transitively and let  $Y \subsetneq X$  be a closed orbit. Now apply Proposition 3.10 to the vanishing ideal of  $Y$ .  $\square$

**3.12. Corollary.** *Let  $H$  be a quasi-elementary group acting on a  $k$ -dense affine variety  $X$ . Then every  $H$ -semiinvariant rational function over  $k$  is the quotient of two  $H$ -semiinvariant regular functions over  $k$ .*

*Proof.* Let  $f \in k(X)^{(H)}$ . Then apply Proposition 3.10 to the ideal  $I = \{h \in k[X] \mid hf \in k[X]\}$ .  $\square$

For reductive groups one can prove the following refinement of Proposition 3.10.

**3.13. Proposition.** *Let  $G$  be a connected reductive group acting on an affine variety  $X$ . Let  $I \subseteq K[X]$  be a non-zero  $G$ -stable ideal and let  $x \in X(k)$  point which is fixed for  $G_{\text{el}}$  and which is not contained in the zero locus of  $I$ . Then  $I$  contains a  $G$ -semiinvariant function  $f \in k[X]$  with  $f(x) \neq 0$ .*

*Proof.* The group  $G$  factorizes uniquely as  $G = MAG_{\text{el}}$  where  $M$  is anisotropic and  $A$  is a split torus. Put  $H := MG_{\text{el}}$ . Then  $Hx = Mx$  is closed because of Corollary 3.7. Moreover, it is disjoint from  $Z$ , the zero locus of  $I$ , since  $x \notin Z$ . Thus there is a function  $F \in K[X]$  which vanishes on  $Z$  and which is non-zero on  $x$ . Replacing  $F$  by the product

of its Galois translates, we may assume that  $F$  is defined over  $k$ . Replacing  $F$  by a suitable power, we may further assume that  $F \in I$ . Since  $A$  is a split torus and  $I$  is  $A$ -stable,  $F$  decomposes as a sum of  $A$ -semiinvariants  $f_{\chi_i} \in I$ . At least one of the  $f_i$  will not vanish in  $x$ . This  $f_i$  has all the asserted properties.  $\square$

If  $P$  is a minimal parabolic subgroup of  $G$  then  $P$  is quasi-elementary and all the results above apply to actions of  $P$ . In case, the  $P$ -action actually comes from a  $G$ -action one can be more specific about the character of a semiinvariant.

**3.14. Lemma.** *Let  $G$  be a connected reductive group acting on an affine  $k$ -dense variety  $X$  and let  $P = MAN$  be a minimal parabolic  $k$ -subgroup of  $G$ . Let  $\chi \in \Xi(A)$  be the character of a  $AN$ -semiinvariant function  $f \in k[X]$ . Then  $\chi^n$  is, for some  $n \in \mathbb{Z}_{>0}$ , the character of a  $P$ -semiinvariant function  $\bar{f} \in k[X]$ .*

*Proof.* We start with a couple of reduction steps. Let  $V_0 \subseteq k[X]$  be the  $AN$ -eigenspace for the character  $\chi$ . Because  $V_0 \neq 0$ , by assumption, it contains an irreducible  $MA$ -submodule  $V_1$ . Then the  $G$ -module  $V := \langle G \cdot V_1 \rangle \subseteq k[X]$  generated by  $V_1$  is irreducible. The inclusion  $V \hookrightarrow k[X]$  yields a  $G$ -morphism  $X \rightarrow V^*$ . Let  $X_1$  be the closure of its image and let  $\ell : V^* \rightarrow k$  be the evaluation in any fixed  $0 \neq f \in V_1$ . Then  $\ell$  is a linear  $AN$ -semiinvariant on  $V^*$  with character  $\chi$  which is non-zero on  $X_1$ . Clearly, it suffices to prove the assertion for  $X_1$ .

Now let  $\bar{X} := \overline{\mathbf{G}_m X_1} \subseteq V^*$  be the closed cone generated by  $X_1$ . It carries an action of  $\bar{G} = \mathbf{G}_m \times G$ . Then  $\bar{P} = M\bar{A}N$  is a minimal parabolic subgroup of  $\bar{G}$  where  $\bar{A} = \mathbf{G}_m \times A$ . Moreover,  $\ell$  is an  $\bar{A}N$ -semiinvariant for the character  $\bar{\chi} = (1, \chi)$ . Now suppose the assertion is true for  $\bar{X}$ . Then  $\bar{X}$  carries a  $\bar{P}$ -semiinvariant  $\bar{f}$  with character  $(1, \chi)^n = (n, \chi^n)$ . Then the restriction  $f_1$  of  $\bar{f}$  to  $X_1$  is non-zero because  $\bar{f} \neq 0$  is homogeneous and  $\mathbf{G}_m X_1$  is dense in  $\bar{X}$ . Then  $f_1$  is the desired semiinvariant.

Replacing  $X, G$  by  $\bar{X}, \bar{G}$  we are reduced to the situation that  $X$  is a closed cone in an irreducible  $G$ -module  $V^*$ , that  $G$  contains all scalars and that  $f$  is the restriction of a linear function  $\ell$  on  $V^*$ .

Let  $N^- \subseteq G$  be the unipotent subgroup which is opposite to  $N$ . Then  $PN^- = MANN^- = NMN^-$  is a dense open subset of  $G$ . Let  $\mathbf{P}(X) \subseteq \mathbf{P}(V^*)$  be the projective variety corresponding to  $X$ . Since  $AN^-$  is a split solvable group and  $X$  is  $k$ -dense it has a fixed point in  $\mathbf{P}(X)$  which means that  $X$  contains an  $AN^-$ -fixed line  $k \cdot w \neq 0$ . Suppose that  $\ell(Mw) = 0$ . Then  $\ell(NMAN^-w) = 0$  and therefore  $\ell(Gw) = 0$  which is impossible since  $V^*$  is irreducible. Thus there is  $g \in M(k)$  with  $\ell(gw) \neq 0$ . Replacing  $w$  by  $gw$  we may assume that  $\ell(w) \neq 0$ . But then  $A$  acts on  $kw$  via the character  $\chi^{-1}$ .

Let  $X_2 := \overline{Gw} \subseteq X$  which is a cone since  $G$  contains the scalars. Because of  $0 \in X_2$ , the group  $MN$  can't act transitively on  $X_2$ . Thus Corollary 3.11 implies that  $X_2$  carries a non-constant  $P$ -semiinvariant  $\tilde{f}$ . Suppose  $\tilde{f}(w) = 0$ . Then, from  $\tilde{f}(PN^-w) = 0$  we get  $\tilde{f}(Gw) = 0$  and therefore  $\tilde{f} = 0$  which is not true. Thus the restriction of  $\tilde{f}$  to  $kw$  is an  $A$ -semiinvariant which is homogeneous of  $n > 0$ . Its character therefore equals  $\chi^n$ . Finally, since the restriction map  $k[X] \rightarrow k[X_2]$  splits as a  $G$ -homomorphism the function  $\tilde{f}$  extends to a  $P$ -semiinvariant  $\bar{f}$  on  $X$  with the same character.  $\square$

**3.15. Remark.** This lemma replaces the integration argument in [KKS15a, Thm. 2.2] (Local Structure Theorem) in the case  $k = \mathbb{R}$ .

As mentioned in the introduction, the main philosophy of this paper is to extend the Borel-Tits theory of reductive groups to actions of reductive groups. We start by defining the analogues of elementary and anisotropic groups.

**3.16. Definition.** Let  $H$  be a connected  $k$ -group acting on a  $k$ -dense variety  $X$ . Then the action (or simply  $X$ ) is called *elementary* or *anisotropic* if the elementary radical  $H_{\text{el}}$  or the anisotropic radical  $H_{\text{an}}$ , respectively, acts trivially on  $X$ .

Clearly, the action is elementary if and only if all  $k$ -rational unipotent elements of  $H$  act trivially on  $X$ . Likewise, the action is anisotropic if  $H_u$  and all split subtori of  $H$  act trivially. Let  $P = MAN \subseteq H$  be a minimal parabolic subgroup. Then it follows from (3.2) or (3.3) that an elementary  $H$ -action is the same as an  $MA$ -action which is trivial on  $MA \cap H_{\text{el}}$ . Similarly, (3.3) implies that an anisotropic  $H$ -action is the same as an  $M$ -action which is trivial on  $M \cap H_{\text{an}}$ . Observe that every anisotropic action is elementary.

Next we give a representation theoretic characterization of elementary actions. It will be the key for the proof of the Local Structure Theorem below.

**3.17. Corollary.** *Assume that the action of the connected reductive group  $G$  on the  $k$ -dense quasi-affine variety  $X$  is not elementary. Then there is a  $P$ -semiinvariant  $f \in k[X]$  whose character  $\chi_f$  does not extend to a character of  $G$ .*

*Proof.* According to [BT65, Prop. 8.4] every  $k$ -rational unipotent element is conjugate to an element of  $N$ . This implies that the action of  $N$  on  $X$  is non-trivial. Therefore  $k[X]$  contains a simple  $G$ -submodule  $V$  with a non-trivial  $N$ -action. Choose an  $AN$ -eigenvector  $f \in V$  and let  $\chi$  be its character. Since  $N$  acts non-trivially, highest weight theory shows that there is a root  $\alpha$  of  $A$  in  $\text{Lie } N$  such that  $\langle \chi, \alpha^\vee \rangle > 0$ . According to Lemma 3.14 there is a  $P$ -semiinvariant  $\bar{f} \in k[X]$  whose character is  $\chi_{\bar{f}} = \chi^n$  for some  $n \in \mathbb{Z}_{>0}$ . Then also  $\langle \chi_{\bar{f}}, \alpha^\vee \rangle > 0$  which implies that  $\chi_{\bar{f}}$  cannot be the restriction of a character of  $G$  to  $A$ .  $\square$

**3.18. Remark.** Using (3.2) it is easy to see that the converse of the corollary holds, as well.

## 4. The local structure theorem

Before we embark in stating and proving the Local Structure Theorem, we need to discuss a regularity property for algebraic group actions.

**4.1. Definition.** Let  $H$  be a linear algebraic group and  $X$  an  $H$ -variety. Then  $X$  is called *locally  $k$ -linear* if for every  $x \in X(k)$  there is an  $H$ -stable open neighborhood  $X_0$ , a finite dimensional representation  $V$  of  $H$ , and an  $H$ -equivariant embedding  $X_0 \hookrightarrow \mathbf{P}(V)$ , where everything is defined over  $k$ .

If  $H$  is connected and  $X$  is normal in every  $k$ -rational point then Sumihiro's theorem [Sum75] implies that  $X$  is locally  $k$ -linear. So the condition is rather mild since at least the normalization of any  $H$ -variety is locally  $k$ -linear. On the other side, local linearity has better functorial properties than normality since  $H$ -stable subvarieties of locally linear varieties are locally linear. In particular, all  $H$ -stable subvarieties of  $\mathbf{P}(V)$  are locally linear.

We proceed with the main tool of this paper. Its purpose is to reduce arbitrary  $G$ -actions to elementary ones.

**4.2. Theorem** (The  $k$ -Local Structure Theorem). *Let  $G$  be a connected reductive  $k$ -group, let  $X$  be a locally  $k$ -linear  $G$ -variety, and let  $Y \subseteq X$  be a  $G$ -stable,  $k$ -dense subvariety. Then there is a parabolic  $k$ -subgroup  $Q \subseteq G$  with Levi decomposition  $Q = LU$  and an  $L$ -stable affine  $k$ -subvariety  $R \subseteq X$  such that:*

- i) *The intersection  $Y_{\text{el}} := R \cap Y$  is a  $k$ -dense affine  $L$ -variety (thus, in particular, non-empty), all  $L$ -orbits in  $Y_{\text{el}}$  are closed, and the action of  $L$  on  $Y_{\text{el}}$  is elementary.*
- ii) *The morphism*

$$(4.1) \quad U \times R = Q \times^L R \rightarrow X : [g, x] \mapsto gx$$

*is an open embedding.*

For the proof, we'll need the following lemma.

**4.3. Lemma.** *Let  $X_0 \subseteq X$  be a  $G$ -stable open  $k$ -subset with  $Y_0 := Y \cap X_0 \neq \emptyset$ . Then the  $k$ -Local Structure Theorem holds for  $(X, Y)$  if and only if it holds for  $(X_0, Y_0)$ .*

*Proof.* One direction is trivial: if the  $k$ -LST holds for  $(X_0, Y_0)$  then it holds for  $(X, Y)$ . Assume conversely, that it holds for  $(X, Y)$  and let  $R \subseteq X$  be the  $L$ -stable slice. Put  $R' := R \cap X_0$ . Because  $U(R \cap Y)$  is dense and open in  $Y$  we see that  $R$  meets every  $U$ -stable open subset of  $Y$ . In particular,  $R' \cap Y_0 \neq \emptyset$ . One readily sees that  $R' \subseteq X_0$  inherits all requirements for a slice from  $R \subseteq X$  except possibly  $R'$  being affine. To force this last property it suffices to construct an affine  $L$ -stable open subset  $R_0 \subseteq R'$  with  $R_0 \cap Y_0 \neq \emptyset$ .

To do this, we choose a  $k$ -rational point  $y \in R \cap Y_0$ . This is possible since  $R \cap Y$  is  $k$ -dense. Then  $y \notin Z := R \setminus R'$ . Because the  $L$ -action on  $Y$  is elementary, the point  $y$  is fixed by  $L_{\text{el}}$ . Then Proposition 3.13 yields an  $L$ -semiinvariant function  $f$  on  $R$  vanishing on  $Z$  and being non-zero in  $y$ . Now  $R_0 := \{x \in R \mid f(x) \neq 0\} \subseteq R'$  is affine open with  $R_0 \cap Y_0 \neq \emptyset$ .  $\square$

*Proof of the  $k$ -LST.* We start with a number of reductions. First, by the lemma and the assumption of local  $k$ -linearity we may assume that  $X \subseteq \mathbf{P}(V)$  where  $V$  is a finite dimensional  $G$ -module. Next we apply the lemma to  $X$  and its closure in  $\mathbf{P}(V)$ . Then we may assume that  $X$  is closed in  $\mathbf{P}(V)$ .

Now let  $\tilde{X} \subseteq V$  and  $\tilde{Y} \subseteq V$  be the affine cones of  $X$  and  $Y$ , respectively. Put moreover  $\tilde{X}^* := \tilde{X} \setminus \{0\}$  and  $\tilde{Y}^* := \tilde{Y} \setminus \{0\}$ . All these varieties carry an action of  $\tilde{G} := G \times \mathbf{G}_m$ . Assume that the  $k$ -LST is true for  $(\tilde{G}, \tilde{X}, \tilde{Y})$ . Then it holds for  $(\tilde{G}, \tilde{X}^*, \tilde{Y}^*)$  by the lemma. Moreover,  $\tilde{X}^* \rightarrow X$  and  $\tilde{Y}^* \rightarrow Y$  are principal  $\mathbf{G}_m$ -bundles. It follows easily, that all assertions of the theorem descend to  $(G, X, Y)$ . Thus, by replacing  $(G, X, Y)$  with  $(\tilde{G}, \tilde{X}, \tilde{Y})$  we may assume that  $X$  is an affine  $G$ -variety.

We proceed by arguing by induction on  $\dim X$ . Assume first that the  $G$ -action on  $Y$  is already elementary. Then  $Q = L = G$  and  $R = X$  have all required properties except possibly the closedness of the  $G$ -orbits in  $Y$ . To force this, let  $Z \subseteq Y$  be the set of all  $y \in Y$  whose orbit  $Gy$  is not of maximal dimension. This is a proper closed  $G$ -stable subset of  $Y$ . Moreover, all  $G$ -orbits in  $Y \setminus Z$  have the same dimension, thus they are

closed. By Proposition 3.13 we can choose a  $G$ -semiinvariant function  $\bar{f}$  on  $Y$  vanishing on  $Z$ . Since  $k[Y]$  is, as a  $G$ -module, a direct summand of  $k[X]$ , the function  $\bar{f}$  extends to a  $G$ -semiinvariant function  $f$  on  $X$ . Let  $X_f$  be the non-vanishing set of  $f$ . Then  $Y_f = Y \cap X_f \neq \emptyset$  and all of its  $G$ -orbits are closed. Thus, we can take  $Q = G$  and  $R = X_f$  and we are done.

Finally, assume that the action of  $G$  on  $Y$  is not elementary. Then Corollary 3.17 implies the existence of a  $P$ -semiinvariant function  $\bar{f} \in k[Y]$  whose character does not extend to  $G$ . As above,  $\bar{f}$  extends to a  $P$ -semiinvariant function  $f \in k[X]$ . Let  $X_f \subseteq X$  be its non-vanishing set. Then  $Y_f = X_f \cap Y = Y_{\bar{f}} \neq \emptyset$ . Let  $\bar{Q} := \{g \in G \mid gf \in k^*f\}$  be the normalizer of  $f$  in  $G$ . Since  $f$  is a highest weight vector,  $\bar{Q}$  is a parabolic subgroup which acts on the line  $kf$  by a dominant character  $\chi_f$ . Moreover, the parabolic  $\bar{Q}$  is a proper subgroup of  $G$  because  $\chi_f$  does not extend to a character of  $G$ . Let  $\bar{U} = \bar{Q}_u$  be its unipotent radical. From highest weight theory we get

$$(4.2) \quad \langle \chi_f, \alpha^\vee \rangle > 0 \text{ for all roots } \alpha \text{ in } \bar{\mathfrak{u}} := \text{Lie } \bar{U}$$

The rest of the argument is almost identical to the proof of [Kno94, Thm. 2.3]. On the  $\bar{Q}$ -stable open set  $X_f$  one can define the  $\bar{Q}$ -equivariant morphism

$$(4.3) \quad m : X_f \rightarrow \mathfrak{g}^* : x \mapsto m_x$$

where

$$(4.4) \quad m_x : \mathfrak{g} \rightarrow K : \xi \mapsto \frac{\xi f(x)}{f(x)}.$$

Choose a  $k$ -rational point  $y \in Y_f$  (possible because of  $k$ -density) and put  $a := m(y) \in \mathfrak{g}^*$ . Then the  $\bar{Q}$ -invariance of  $f$  implies that the image of  $m$  is contained in the affine subspace

$$(4.5) \quad a + \bar{\mathfrak{q}}^\perp = a + \bar{\mathfrak{u}}$$

where in the last equality we used an isomorphism  $\mathfrak{g}^* \cong \mathfrak{g}$  to identify  $\bar{\mathfrak{q}}^\perp$  with  $\bar{\mathfrak{u}}$ .

Next, it follows from (4.2) that the stabilizer of  $a$  in  $G$  is a Levi subgroup  $\bar{L}$  of  $\bar{Q}$ . It follows moreover, that  $a + \bar{\mathfrak{u}} = \bar{Q}a = \bar{Q}/\bar{L}$  is a single  $\bar{Q}$ -orbit. In conclusion, we got a  $\bar{Q}$ -equivariant morphism  $m : X_f \rightarrow \bar{Q}/\bar{L}$ . This means that if we define  $X' := m^{-1}(a)$  then

$$(4.6) \quad \bar{U} \times X' = \bar{Q} \times^{\bar{L}} X' \xrightarrow{\sim} X_f : [q, x] \mapsto qx.$$

is an isomorphism.

Because  $\bar{U} \neq 1$  we have  $\dim X' < \dim X$ . Therefore, we can apply the induction hypothesis to  $\bar{L}$  acting on  $X'$  and  $Y' := X' \cap Y$ . This yields a parabolic  $k$ -subgroup  $Q' \subseteq \bar{L}$ , a Levi decomposition  $Q' = L'U'$ , and an  $L'$ -stable affine slice  $R \subseteq X'$  such that

$$(4.7) \quad U' \times R = Q' \times^{L'} R \rightarrow X'$$

is an open immersion. Now put  $Q := Q'\bar{U} \subseteq \bar{L}\bar{U} \subseteq G$ , a parabolic subgroup of  $G$ . Then combining (4.6) and (4.7) we get that

$$(4.8) \quad Q \times^L R = Q_u \times R = \bar{U} \times U' \times R \hookrightarrow \bar{U} \times X' \hookrightarrow X$$

is an open immersion. □

Since  $Q \times^L R = U \times R$  is affine we get as an immediate consequence:

**4.4. Corollary.** *Let  $Y$  be a  $G$ -stable  $k$ -dense subvariety of a locally  $k$ -linear  $G$ -variety  $X$ . Then there is a  $P$ -stable affine open  $k$ -subset  $X_0 \subseteq X$  with  $X_0 \cap Y \neq \emptyset$ .*

The slice  $R$  is not unique, but we still have some uniqueness properties. First, we define for any  $k$ -dense  $G$ -variety  $X$ :

$$(4.9) \quad Q_k(X) := \{g \in G \mid gPx = Px \text{ for } x \text{ in a dense open subset of } X\}.$$

Because of  $P \subseteq Q_k(X)$ , this is a parabolic  $k$ -subgroup.

**4.5. Proposition.** *In the setting of Theorem 4.2 assume (without loss of generality) additionally that  $P \subseteq Q$ . Then  $Q = Q_k(Y)$ . Moreover, as an  $L$ -variety, the slice  $R$  is unique up to a unique  $L$ -equivariant birational isomorphism which is regular in a generic point of  $Y_{\text{el}}$ .*

*Proof.* Put, for the moment,  $\overline{Q} = Q_k(Y)$ . From (3.2) and the  $k$ -LST follows that the generic  $P$ -orbits and  $Q$ -orbits on  $Y$  coincide. This shows  $Q \subseteq \overline{Q}$ .

For the converse, let  $y \in Y_{\text{el}}$  be generic. Then  $\overline{Q}y = Py$  by assumption and  $Py = Qy$  by the  $k$ -LST. Thus  $\overline{Q}/\overline{Q}_y = Q/Q_y$  which implies  $Q_y = Q \cap \overline{Q}_y$  and  $\overline{Q} = Q\overline{Q}_y$ . This in turn implies  $\overline{Q}_y/Q_y = \overline{Q}/Q$ . Now observe that  $Q_y = L_y$  is reductive since the  $L$ -action on  $Y$  is elementary. Therefore  $\overline{Q}_y/Q_y$  is an affine variety. On the other hand,  $\overline{Q}/Q$  is projective which implies  $\overline{Q} = Q$ .

Finally, let  $X_0 = QR \subseteq X$ . This is an open, dense,  $Q$ -stable subset with  $X_0 \cap Y \neq \emptyset$  such that the orbit space  $X_0/Q_u$  exists. Moreover,  $R$  is isomorphic to this quotient. If  $X_1 \subseteq X$  is any other such set then  $R$  is birationally equivalent to  $(X_0 \cap X_1)/Q_u$  and therefore to  $X_1/Q_u$ . This shows the second assertion.  $\square$

One of the most important special cases is that of  $Y = X$ . We formulate it explicitly:

**4.6. Corollary.** *Let  $X$  be a  $k$ -dense  $G$ -variety and let  $Q_k(X) = LU$  be a Levi decomposition. Then there exists a smooth affine  $L$ -subvariety  $X_{\text{el}} \subseteq X$  such that*

- i) the action of  $L$  on  $X_{\text{el}}$  is elementary, all orbits are closed, and the categorical quotient  $X_{\text{el}} \rightarrow X_{\text{el}}//L$  is a locally trivial fiber bundle in the étale topology.*
- ii) the natural morphism  $U \times X_{\text{el}} = Q_k(X) \times^L X_{\text{el}} \rightarrow X$  is an open embedding.*

*The slice  $X_{\text{el}}$  is unique up to a unique  $L$ -equivariant birational isomorphism. More precisely, its field of rational functions can be computed as*

$$(4.10) \quad k(X_{\text{el}}) = k(X)^U = k(X)^N.$$

*Proof.* Apply Theorem 4.2 to the smooth part of  $X$  which is locally linear by Sumhiro's theorem. This proves everything except for the local triviality statement and the second equality in (4.10). The former is well known consequence of Luna's slice theorem (see [Lun73, III.2, Cor. 5]). The latter is a consequence of the fact that  $N = U \cdot (L \cap N)$  and that  $L \cap N$  acts trivially on  $X_{\text{el}}$ . Hence the generic  $N$ - and  $U$ -orbits coincide on  $X$ .  $\square$

Because of the factorization (3.2) we don't lose any information by considering  $X_{\text{el}}$  as an  $MA$ -variety. Since all  $A$ -orbits in  $X_{\text{el}}$  are closed we may form the orbit space.

**4.7. Definition.** The quotient  $X_{\text{an}} := X_{\text{el}}//A$  is called the *anisotropic kernel* of  $X$ .

The anisotropic kernel is, by construction a  $k$ -dense affine  $M$ -variety all of whose orbits are closed of the same dimension. In particular, all isotropy groups are reductive and, over  $K$ , conjugate to each other. In our theory  $X_{\text{an}}$  will play a role similar to the anisotropic kernel of a reductive group in Borel-Tits theory [BT65]. This means that the anisotropic kernel will mostly serve as a black box.

From (4.10) we see that

$$(4.11) \quad k(X_{\text{an}}) = k(X)^{AN} \text{ and therefore } k(X_{\text{an}})^M = k(X)^P.$$

This provides another way to see that the anisotropic kernel is a birational invariant for  $X$ .

Next we introduce an important numerical invariant for the action of  $G$  on  $X$ . Recall that the *cohomogeneity* of an action is the codimension of a generic orbit.

**4.8. Definition.** The  $k$ -complexity of  $X$ , denoted  $c_k(X)$ , is the cohomogeneity for the  $P$ -action on  $X$ . It also equals the transcendence degree of  $k(X)^P$  (by Rosenlicht), the transcendence degree of  $k(X_{\text{an}})^M$ , and the cohomogeneity of the  $M$ -action on  $X_{\text{an}}$ .

The most important special case is:

**4.9. Definition.** A locally linear,  $k$ -dense  $G$ -variety  $X$  with  $c_k(X) = 0$  is called  $k$ -spherical. Equivalently,  $X$  is  $k$ -spherical if it is locally linear,  $k$ -dense and  $P$  has an open orbit.

Another characterizing property for  $X$  being  $k$ -spherical is that  $M$  acts transitively on the anisotropic kernel. Observe that in this case  $X_{\text{an}}$  is even unique up to a biregular (as opposed to a birational) isomorphism.

**4.10. Remarks.** *i)* Let  $B \subseteq P$  be a Borel subgroup. Then  $X$  is  $K$ -spherical if  $B$  has a dense orbit in  $X$ , i.e., if  $X$  is spherical in the usual sense. In this case, we call  $X$  *absolutely spherical*. Clearly all absolutely spherical varieties are  $k$ -spherical. In particular, a symmetric variety  $G/H$  with  $H$  the fixed point group of an involution is  $k$ -spherical. The arguably most important example is that of  $G/K$  where  $k = \mathbb{R}$  and  $K$  is a maximal compact subgroup of  $G$ .

*ii)* A  $k$ -variety is absolutely spherical if and only if its anisotropic kernel is absolutely spherical. Over  $k = \mathbb{R}$ , the latter have been classified in works of Krämer [Krä79], Mikityuk [Mik86], and Brion [Bri87].

*iii)* If  $G$  is quasi-split then  $P = B$ . Thus, every  $k$ -spherical variety is absolutely spherical. Otherwise,  $X = G/AN$  is an example of a  $k$ -spherical variety which is not absolutely spherical. Its anisotropic kernel is  $MA/A = M/M \cap A$ .

For quasi-affine varieties there is a representation theoretic criterion which was first proved over algebraically closed fields by Vinberg-Kimel'feld [VK78].

**4.11. Proposition.** *For a quasi-affine,  $k$ -dense  $G$ -variety  $X$  the following are equivalent:*

- i)*  $X$  is  $k$ -spherical.
- ii)* For every dominant character  $\chi$  of  $P$ , the simple  $G$ -module with highest weight  $\chi$  appears with multiplicity at most 1 in  $k[X]$ .

*Proof.* Let  $M_1, M_2 \subseteq k[X]$  two distinct simple  $G$ -submodules with highest weight  $\chi$ . Let  $f_i \in M_i$  be a highest weight vector. Then  $f = \frac{f_1}{f_2}$  is a non-constant  $P$ -invariant rational function on  $X$ . Thus  $X$  cannot have an open  $P$ -orbit.

Conversely, assume that  $X$  does not have an open  $P$ -orbit. Then by Rosenlicht there is a non-constant  $P$ -invariant rational function on  $X$ . By Corollary 3.12, there are  $P$ -semiinvariants  $f_1, f_2 \in k[X]$  with  $f = \frac{f_1}{f_2}$ . The character of  $f_1$  and  $f_2$  are the same, say  $\chi$ . Hence the  $G$ -modules  $M_i$  generated by  $f_i$  are distinct and irreducible with highest weight  $\chi$ .  $\square$

For  $k$ -spherical varieties Corollary 4.6 reads as follows:

**4.12. Corollary.** *Let  $X$  be a  $k$ -spherical variety  $G$ -variety and  $Q := Q_k(X) = LU$  be a Levi decomposition. Then there is a point  $x \in X(k)$  such that  $Qx$  is open and affine in  $X$  and the isotropy subgroup satisfies  $L_{\text{el}} \subseteq Q_x \subseteq L$ .*

Next we compare the complexity of a variety with that of its subvarieties.

**4.13. Proposition.** *Let  $X$  be a locally linear  $k$ -dense  $G$ -variety and let  $Y \subseteq X$  be a  $G$ -stable  $k$ -dense subvariety. Then  $c_k(Y) \leq c_k(X)$  with equality if and only if  $K(X)^P \subseteq \mathcal{O}_{X,Y}$  (the local ring of  $X$  in  $Y$ ).*

*Proof.* We apply Theorem 4.2 to  $Y \subseteq X$ . Since all  $L$ -orbits in  $Y_{\text{el}}$  are closed, the morphism

$$(4.12) \quad Y_{\text{el}}//L \rightarrow R//L$$

is a closed embedding. Thus

$$(4.13) \quad c_k(Y) = c_k(Y_{\text{el}}) = \dim Y_{\text{el}}//L \leq \dim R//L \leq c_k(R) = c_k(X).$$

In case of equality, (4.12) is an isomorphism. From

$$(4.14) \quad K(X)^P \subseteq K(X)^{LU} = K(R)^L = K(Y_{\text{el}})^L = K(Y)^{LU} \subseteq K(Y).$$

we see that all  $P$ -invariant rational functions are regular in  $Y$ . Conversely,  $K(X)^P \subseteq K(Y)$  clearly implies  $c_k(Y) \geq \text{trdeg } K(X)^P = c_k(X)$ .  $\square$

**4.14. Corollary.** *Let  $X$  be a  $k$ -spherical  $G$ -variety. Then every  $k$ -dense  $G$ -stable subvariety is  $k$ -spherical and  $X$  contains only finitely many of them. In particular, the number of  $G$ -orbits  $Y$  with  $Y(k) \neq \emptyset$  is finite.*

*Proof.* The first assertion follows directly from Proposition 4.13. Suppose  $X$  contains infinitely many  $k$ -dense subvarieties. Then there is a minimal closed  $G$ -stable subvariety  $Y$  with the same property. Since  $Y$  is spherical,  $P$  and therefore  $G$  has a dense open orbit  $Y^0$ . But then one of the irreducible components of  $Y \setminus Y^0$  would be an even smaller counterexample.  $\square$

**4.15. Corollary.** *Let  $k$  be a local field and  $X$  a  $k$ -spherical variety. Then the number of  $G(k)$ -orbits in  $X(k)$  is finite.*

*Proof.* This follows from Corollary 4.15 and [BS64, Cor. 6.4]: if  $k$  is local (of characteristic 0) then for any homogeneous  $G$ -variety  $Y$  the orbit set  $Y(k)/G(k)$  is finite.  $\square$



Next we discuss the relationship between  $X_{\text{el}}$  and  $X_{\text{an}}$ . Let  $A_0 \subseteq A$  be the kernel of the action of  $A$  on  $X_{\text{el}}$ . Then  $A_k = A_k(X) := A/A_0$  is acting freely on  $X_{\text{el}}$  and the quotient map

$$(4.15) \quad \pi : X_{\text{el}} \rightarrow X_{\text{an}}$$

is an  $M$ -equivariant principal bundle for  $A_k$ . We make this more precise. The character group of  $A_k$  consists of the characters  $\chi_f \in \Xi(A)$  where  $f$  is an  $A$ -semiinvariant rational function on  $X_{\text{el}}$ . The Local Structure Theorem implies that these  $f$  correspond to  $AN$ -semiinvariant rational functions on  $X$ . From this, we derive another fundamental invariant of  $X$ :

$$(4.16) \quad \Xi(A_k(X)) = \Xi_k(X) := \{\chi_f \in \Xi(A) \mid f \in K(X)^{(AN)}\}.$$

Since  $\chi_f = 1$  if  $f$  is  $AN$ -invariant, the group  $\Xi_k(X)$  fits into the short exact sequence

$$(4.17) \quad 1 \rightarrow K(X_{\text{an}})^* \rightarrow K(X)^{(AN)} \rightarrow \Xi_k(X) \rightarrow 0.$$

**4.16. Definition.** Let  $X$  be a  $k$ -dense  $G$ -variety. The  $k$ -rank of  $X$  is

$$(4.18) \quad \text{rk}_k X := \text{rk } \Xi_k(X) = \dim A_k(X) = \dim X_{\text{el}} - \dim X_{\text{an}}.$$

Coming back to (4.15), the algebra  $k[X_{\text{el}}]$  decomposes as an  $A_k$ -module into isotypic components:

$$(4.19) \quad k[X_{\text{el}}] = \bigoplus_{\chi \in \Xi_k(X)} \mathcal{S}_\chi.$$

Moreover, each  $\mathcal{S}_\chi$  is an  $M$ -line bundle over  $\mathcal{R} := \mathcal{S}_0 = k[X_{\text{an}}]$ . Since  $\mathcal{S}_\chi \otimes_{\mathcal{R}} \mathcal{S}_\eta = \mathcal{S}_{\chi+\eta}$  this induces a homomorphism

$$(4.20) \quad \Xi_k(X) \rightarrow \text{Pic}^M(X_{\text{an}}) : \chi \mapsto [\mathcal{S}_\chi]$$

which encodes the non-triviality of the bundle (4.15).

If  $X$  is  $k$ -spherical the situation simplifies. After fixing a point  $y \in X_{\text{an}}(k)$  we get  $X_{\text{an}} \cong M/M_y$ . Then for any  $m \in M_y$  there is a unique  $a \in A_k$  with  $mx = ax$  for all  $x \in \pi^{-1}(y)$ . This way, we get a homomorphism

$$(4.21) \quad \varphi : M_y \rightarrow A_k : m \mapsto a$$

which is defined over  $k$ . Using  $\varphi$ , one can recover  $X_{\text{el}}$  as

$$(4.22) \quad X_{\text{el}} \cong M \times^{M_y} A_k$$

**4.17. Remarks.** *i)* Since the connected component  $M_y^\circ$  is anisotropic,  $\varphi$  is trivial on it. Thus  $\varphi$  factors through the finite group  $\pi_0(M_y)$ . In particular,  $\varphi(M_y)$  is finite.

*ii)* Besides  $\Xi_k(X)$  it seems natural to consider the group

$$(4.23) \quad \Xi'_k(X) := \{\chi_f \in \Xi(A) \mid f \in K(X)^{(P)}\}.$$

It is easy to see that  $\Xi'_k(X)$  is the character group of the torus  $A'_k(X) := A_k(X)/\varphi(M_y)$ . We won't have use for this group, though.

We finish this section by stating a comparison result between the theories over  $k$  and  $K$ . For this, choose a maximal torus  $T \subseteq MA$  which is defined over  $k$ . Let, moreover  $B$  be a Borel subgroup of  $P$  (and therefore of  $G$ ) containing  $T$ . Unless  $G$  is quasi-split,  $B$  won't be defined over  $k$ . The torus  $A$  is the maximal split subtorus of  $T$ . Recall the restriction map  $\text{res}_A : \Xi(T) \rightarrow \Xi(A)$ .

**4.18. Lemma.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then*

$$(4.24) \quad \Xi_k(X) = \text{res}_A \Xi_K(X).$$

*Proof.* The inclusion “ $\supseteq$ ” follows from the fact that every  $B$ -semiinvariant is also an  $AN$ -semiinvariant for the restricted character. We show the opposite inclusion. Since  $\Xi_k(X) = \Xi_k(X_{\text{el}})$  it suffices to prove the statement for  $MA$  acting on  $X_{\text{el}}$ . Since all  $A$ -orbits in  $X_{\text{el}}$  are closed, every rational  $A$ -semiinvariant on  $X_{\text{el}}$  is the quotient of two regular  $A$ -semiinvariants. Thus it suffices to show that for every  $f \in K[X_{\text{el}}]^{(A)}$  there is  $\bar{f} \in K[X_{\text{el}}]^{(B_0)}$  with  $\chi_f = \text{res}_A \chi_{\bar{f}}$  where  $B_0 := B \cap MA$ . For this consider the  $A$ -eigenspace  $\mathcal{S}_\chi := K[X_{\text{el}}]_\chi$ . This space is  $MA$ -stable and non-zero by assumption. Then any highest weight vector  $\bar{f} \in \mathcal{S}_\chi$  will be a  $B_0$ -semiinvariant with the required property.  $\square$

This result can be rephrased. By construction, the tori  $A_k(X)$  and  $A_K(X)$  are quotients of  $A$  and  $T$ , respectively. Then the lemma means:

**4.19. Corollary.** *The torus  $A_k(X)$  is the image of  $A \subseteq T$  in  $A_K(X)$ .*

A slightly weaker result goes as follows. Recall the notation  $\mathcal{N}(A) = \text{Hom}(\Xi(A), \mathbb{Q})$  and put

$$(4.25) \quad \mathcal{N}_k(X) = \mathcal{N}(A_k(X)) = \text{Hom}(\Xi_k(X), \mathbb{Q}).$$

This is a  $\mathbb{Q}$ -vector space of dimension  $\text{rk}_k X$ .

**4.20. Corollary.** *The space  $\mathcal{N}_k(X)$  is the image of  $\mathcal{N}(A) \subseteq \mathcal{N}(T)$  in  $\mathcal{N}_K(X)$ .*

**4.21. Remarks.** 1. Observe that  $T$  is not unique, not up to conjugacy nor even up to isomorphism. An exception is the field  $k = \mathbb{R}$  of real numbers where  $T$  is unique up to conjugation by  $M(\mathbb{R})$ .

2. The character group  $\Xi_k(X)$  can also be defined as

$$(4.26) \quad \Xi_k(X) := \{\chi_f \in \Xi(A) \mid f \in k(X)^{(AN)}\},$$

i.e., with semiinvariants defined over  $k$  instead of  $K$ . To see this, one can again reduce to the case that  $G$  is elementary,  $X$  is affine and  $f$  is regular. The point is now that since  $A$  is split, the character  $\chi_f$  is defined over  $k$  even if  $f$  isn't. Then  $k[X]_{\chi_f} \otimes_k K \cong K[X]_{\chi_f} \neq 0$  implies  $k[X]_{\chi_f} \neq 0$ .

**4.22. Example.** We illustrate the theory with a simple but instructive example. Let  $D$  be a finite dimensional division algebra over  $k$  whose center is denoted by  $E$ . For  $n \geq 2$  let

$$(4.27) \quad H_1 := 1 \times GL(n-1, D) \subset GL(1, D) \times GL(n-1, D) \subseteq G := GL(n, D).$$

Then  $X := G/H_1$  can be realized as follows: let  $V = D^n$  be the space of column vectors with  $G$ -action on the left,  $g \cdot \underline{v} := g\underline{v}$  and  $V^\top$  be the space of row vectors with action  $g \cdot \underline{u}^\top := \underline{u}^\top g^{-1}$ . Then the map  $b : V^\top \times V \rightarrow D$  with  $b(\underline{u}^\top, \underline{v}) = \sum_i u_i v_i$  is  $G$ -invariant. Since  $H_1$  is the isotropy group of  $(\underline{e}_1^\top, \underline{e}_1)$ , we get

$$(4.28) \quad X = \{(\underline{u}^\top, \underline{v}) \in V^\top \times V \mid b(\underline{u}^\top, \underline{v}) = 1_D\}.$$

Let  $Q \subseteq G$  be the parabolic subgroup fixing the subspaces  $D\underline{e}_1 \subseteq V$  and  $\underline{e}_n^\top D \subseteq V^\top$ . Then  $Q = LU$  with

$$(4.29) \quad L = GL(1, D) \times GL(n-2, D) \times GL(1, D)$$

and  $U \cong D^{2n-3}$  (as a space). Let  $X_0 \subseteq X$  be the open set of  $(\underline{u}^\top, \underline{v})$  with  $\underline{u}_1, \underline{v}_n \in D^\times = GL(1, D)$ . Then it is easy to see that  $U$  acts freely on  $X_0$  and that the map

$$(4.30) \quad \pi : X_0 \rightarrow D^\times \times D^\times : (\underline{u}^\top, \underline{v}) \mapsto (u_1, v_n)$$

is a quotient by  $U$ . This shows that  $Q_k(X) = Q$  and  $X_{\text{el}} = D^\times \times D^\times$  with  $L$ -action

$$(4.31) \quad (d_1, d_2, d_3)(u_1, v_n) = (u_1 d_1^{-1}, d_3 v_n).$$

Since that action is transitive and elementary we see that  $X$  is  $k$ -spherical with  $\text{rk}_k X = 2$ . The anisotropic kernel is  $X_{\text{an}} = PGL(1, D) \times PGL(1, D)$ . On the other side,  $X_{\text{an}}$  is  $K$ -spherical if and only if  $D = E$  is a field.

## 5. Invariant valuations

Let  $X$  initially be any  $K$ -variety. In this section we study a particular type of valuations of the field  $K(X)$  of  $K$ -valued rational functions on  $X$ . More precisely, for us, a valuation is a map  $v : K(X) \rightarrow \mathbb{Q} \cup \{\infty\}$  satisfying

- i)  $v(f_1 f_2) = v(f_1) + v(f_2)$ ,
- ii)  $v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\}$ ,
- iii)  $v(f) = \infty \Leftrightarrow f = 0$ ,
- iv)  $v(K^*) = 0$ ,

where  $f, f_1, f_2$  run through all elements of  $K(X)$ . Note, that we also allow the trivial valuation  $v = 0$  with  $v(f) = 0$  for all  $f \neq 0$ . Two valuations  $v_1, v_2$  will be called *equivalent* ( $v_1 \sim v_2$ ) if there is  $c \in \mathbb{Q}_{>0}$  with  $v_1 = c v_2$ .

If  $X$  is normal then any irreducible divisor  $D \subset X$  induces a valuation  $v_D$  by order of vanishing along  $D$ . More generally, let  $\overline{X}$  be a normal variety which is birational to  $X$ . Since  $K(\overline{X}) = K(X)$ , any irreducible divisor  $D \subset \overline{X}$  induces a valuation  $v_D$  of  $K(X)$ . Valuations which are equivalent to some  $v_D$  are called *geometric*. It is useful to allow  $D = X$  with  $v_X := 0$  and call it geometric, as well. Geometric valuations are characterized by the property that the transcendence degree of their residue field is  $\geq \dim X - 1$ .

A pair  $(\overline{X}, D)$  with  $v \sim v_D$  is called a *model for  $v$* . The model is unique in the following sense: if  $X', D'$  is another model for  $v$  then the composition of birational maps  $\overline{X} \dashrightarrow X \dashrightarrow X'$  is regular in a generic point of  $D$ , its inverse is regular in a generic point of  $D'$  and it sends  $D$  to  $D'$ .

If  $X$  is defined over  $k$  then it is easy to see that  $\overline{X}$  can be chosen to be defined over  $k$ , as well. If additionally  $v$  is invariant for the action of the Galois group then one can achieve that  $D$  is defined over  $k$ , as well.

Now assume that a connected linear algebraic group  $H$  acts on  $X$ . Then  $H$  (or rather its group  $H(K)$  of rational points) acts on the set of valuations of  $X$ . In particular,  $v$  is  $H$ -invariant if

$$(5.1) \quad v(hf) = v(f) \text{ for all } h \in H \text{ and } f \in K(X).$$

An infinitesimal version goes as follows: the Lie algebra  $\mathfrak{h}$  of  $H$  acts on  $K(X)$  by derivations. Then a valuation  $v$  is  $H$ -invariant if and only if

$$(5.2) \quad v(\xi f) \geq v(f) \text{ for all } \xi \in \mathfrak{h} \text{ and } f \in K(X)$$

(see [LV83, 3.2. Cor. 3]).

The correspondence between divisors and valuations works also in the equivariant setting: if  $v$  is an  $H$ -invariant geometric valuation of  $K(X)$  then there is a normal model  $\overline{X}$  of  $X$  equipped with an  $H$ -action and an  $H$ -stable irreducible divisor  $D \subset \overline{X}$  such that  $v \sim v_D$  ([Kno93, Kor. 7.2]). Of particular importance is the case when  $X$  is a homogeneous variety. Then an equivariant model of  $X$  is the same as an open embedding  $X \hookrightarrow \overline{X}$ , a so-called *equivariant embedding of  $X$* . Moreover, any invariant irreducible divisor must be an irreducible component of the boundary  $\overline{X} \setminus X$ .

In this paper, only a very special kind of valuations is of interest for us.

**5.1. Definition.** An invariant valuation of a  $k$ -dense  $G$ -variety  $X$  is called  *$k$ -central* if it is trivial on the subfield  $K(X_{\text{an}}) = K(X)^{AN}$ . The set of all  $k$ -central valuations is denoted by  $\mathcal{Z}_k(X)$ .

It follows directly from the definition and the short exact sequence (4.17) that a  $k$ -central valuation induces a homomorphism

$$(5.3) \quad \lambda_v : \Xi_k(X) \rightarrow \mathbb{Q} : \chi_f \mapsto v(f),$$

from which we get a map

$$(5.4) \quad \iota_k : \mathcal{Z}_k(X) \rightarrow \mathcal{N}_k(X).$$

Over  $K$ , central valuations have been extensively studied in [Kno93]. For transferring properties from  $K$  to  $k$  we choose  $T \subseteq B \subseteq P$  as in the end of section 4. Recall (see Corollary 4.20) that  $\mathcal{N}_k(X)$  can be considered as a subspace of  $\mathcal{N}_K(X)$ .

**5.2. Proposition.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then:*

- i) *The map  $\iota_k$  in (5.4) is injective.*
- ii) *Considering  $\mathcal{Z}_k(X)$  and  $\mathcal{Z}_K(X)$  as subsets of  $\mathcal{N}_k(X)$  and  $\mathcal{N}_K(X)$ , respectively, then*

$$(5.5) \quad \mathcal{Z}_k(X) = \mathcal{Z}_K(X) \cap \mathcal{N}_k(X).$$

*Proof.* From  $AN \subseteq B$  it follows that  $K(X)^{AN} \supseteq K(X)^B$ . Thus, every  $k$ -central valuation is also  $K$ -central. From this, we get the commutative diagram

$$(5.6) \quad \begin{array}{ccc} \mathcal{Z}_k(X) & \hookrightarrow & \mathcal{Z}_K(X) \\ \downarrow \iota_k & & \downarrow \iota_K \\ \mathcal{N}_k(X) & \hookrightarrow & \mathcal{N}_K(X) \end{array} .$$

Hence  $\iota_k$  is injective since  $\iota_K$  is injective by [Kno93, Kor. 3.6].

It remains to be shown that  $\mathcal{Z}_K(X) \cap \mathcal{N}_k(X) \subseteq \mathcal{Z}_k(X)$ . Let  $v$  be in the intersection. Then  $v$  is trivial on all  $B$ -semiinvariants  $f$  with  $\text{res}_A \chi_f = 0$ , i.e., on  $(K(X)^{(B)})^A$ . Now we have

$$(5.7) \quad (K(X)^{(B)})^A = ((K(X)^{AN})^{(M \cap B)}) = K(X_{\text{an}})^{(M \cap B)}.$$

It follows again from [Kno93, Kor. 3.6]) that  $v$  is trivial on  $K(X_{\text{an}})$ . Thus  $v$  is  $k$ -central.  $\square$

From this we get:

**5.3. Corollary.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then  $\mathcal{Z}_k(X)$  is a finitely generated convex cone in  $\mathcal{N}_k(X)$ .*

*Proof.* This follows from (5.5) and the corresponding statement for  $\mathcal{Z}_K(X)$  (see [Kno93, Kor. 6.5]).  $\square$

In principle,  $\mathcal{Z}_k(X)$  could be still reduced to a point. We show that it is actually fairly big. For this let

$$(5.8) \quad \mathcal{N}^-(A) = \{a \in \mathcal{N}(A) \mid \alpha(a) \leq 0 \text{ for all } \alpha \in S_k\}$$

be the antidominant Weyl chamber with respect to the restricted root system of  $G$ .

**5.4. Proposition.** *Let  $\pi : \mathcal{N}(A) \rightarrow \mathcal{N}_k(X)$  be the canonical projection. Then*

$$(5.9) \quad \pi(\mathcal{N}^-(A)) \subseteq \mathcal{Z}_k(X).$$

*In particular, the interior of  $\mathcal{Z}_k(X)$  is non-empty.*

*Proof.* Let  $\mathcal{N}^-(T)$  be the antidominant Weyl chamber of  $G$  over  $K$ . Then  $\mathcal{N}^-(A) = \mathcal{N}^-(T) \cap \mathcal{N}(A)$ . By [Kno93, Kor. 5.2.4], the image of  $\mathcal{N}^-(T)$  in  $\mathcal{N}_K(X)$  is contained in  $\mathcal{Z}_K(X)$ . Thus,  $\mathcal{Z}_k(X)$  contains the image of  $\mathcal{N}^-(A)$ .  $\square$

## 6. Valuations and geometry

Next, we study the generic structure of invariant divisors defined by valuations. For this, we need a refinement of Corollary 4.6 which also works “at infinity”.

To set this up, let  $G$  be a connected reductive group and let  $X$  be a normal  $k$ -dense  $G$ -variety. According to Corollary 4.6 there is an open embedding  $U \times X_{\text{el}} \hookrightarrow X$  with  $U := Q_k(X)_u$ . By projection to the first factor we get a dominant rational map  $u : X \dashrightarrow U$  with the characterizing property that  $u(x)^{-1}x \in X_{\text{el}}$  for generic  $x \in X$ . This map induces a pull-back homomorphism of function fields  $u^* : K(U) \hookrightarrow K(X)$ .

**6.1. Lemma.** *Let  $v$  be a  $G$ -invariant valuation of a  $k$ -dense  $G$ -variety  $X$ . Then  $v(u^*(h)) \geq 0$  for all regular functions  $h \in K[U]$ .*

*Proof.* The proof depends on the explicit construction of  $X_{\text{el}}$  in the proof of Theorem 4.2. First, we constructed a  $G$ -stable open subset  $X_0 \subseteq X$  which is embeddable in some  $\mathbf{P}(V)$  and considered the affine cone  $\tilde{X} \subseteq V$  over  $X_0$ . We claim that it suffices to prove our assertion for  $\tilde{X}$  and the group  $\tilde{G} = \mathbf{G}_m \times G$ . First of all observe that the slice  $\tilde{X}_{\text{el}} \subseteq \tilde{X}$  is homogeneous. This implies that the projection  $\tilde{u} : \tilde{X} \dashrightarrow U$  factors through  $u : X \dashrightarrow U$ . Secondly, it is known from [Kno93, Kor. 3.2] that the valuation  $v$  on  $X$  can be lifted to a  $\mathbf{G}_m \times G$ -invariant valuation  $\tilde{v}$  on  $\tilde{X}$ . Then

$$(6.1) \quad v(u^*(h)) = \tilde{v}(\tilde{u}^*(h)) \geq 0$$

proves the claim. Replacing  $X$  by  $\tilde{X}$  we may assume that  $X$  is a subvariety of  $V$ .

Next, we used a regular  $P$ -semiinvariant regular function  $f$  on  $X$ . The stabilizer of the line  $Kf$  is the parabolic  $\bar{Q} = \bar{L}\bar{U}$ . Then a  $\bar{Q}$ -equivariant map  $m : X_f \dashrightarrow a + \bar{u}$  with fiber  $X' := m^{-1}(a)$  was defined such that  $\bar{t} : \bar{U} \times X' \xrightarrow{\sim} X_f$ . Finally, application of the induction hypothesis yielded an open embedding  $\iota' : U' \times R \hookrightarrow X'$ .

The open embeddings  $\bar{\iota}$  and  $\iota'$  allow us to define projections  $\bar{u} : X \dashrightarrow \bar{U}$  and  $u' : X' \dashrightarrow U'$  onto the first factors. Then  $\pi(x) := \bar{u}(x)^{-1}x$  is the projection of  $X$  to the second factor  $X'$ . With this notation,  $u$  has the form

$$(6.2) \quad u : X \dashrightarrow U = \bar{U} \times U' : x \mapsto (\bar{u}(x), u'(\pi(x)))$$

Clearly, it is enough to prove the non-negativity of  $v(u^*(h))$  for generators  $h$  of the algebra  $K[U]$ . Thus, we may assume that  $h$  is either a function on  $\bar{U}$  or on  $U'$ . In the first case observe that  $\bar{u}$  is the composition of  $m : X_f \rightarrow a + \bar{u}$  with the isomorphisms  $a + \bar{u} \cong \bar{U}$ . The functions on the affine space  $a + \bar{u}$  are generated by the linear functions  $\ell_\xi := (\xi, *)$  with  $\xi \in \mathfrak{g}$  where  $(\cdot, \cdot)$  is an invariant scalar product on  $\mathfrak{g}$ . From the definition of  $m$  we get

$$(6.3) \quad v(m^*(\ell_\xi)) = v\left(\frac{\xi f}{f}\right) = v(\xi f) - v(f).$$

Now the  $G$ -invariance of  $v$  implies that the right-hand side is non-negative (see (5.2)).

Finally assume that  $h$  is a regular function on  $U'$ . We restrict the valuation  $v$  via  $\pi$  to  $X'$ , i.e., for any  $h' \in K(X')$  we define  $v'(h') := v(\pi^*(h'))$ . It easily seen that  $v'$  is an  $\bar{L}$ -invariant valuation of  $X'$ . The induction hypothesis then yields

$$(6.4) \quad v((u' \circ \pi)^*(h)) = v'((u')^*(h)) \geq 0. \quad \square$$

As a consequence we get the existence of very big slices. For that we have to give up the affinity of the slice and restrict to normal varieties.

**6.2. Theorem.** *Let  $X$  be a normal  $k$ -dense  $G$ -variety and let  $Q_k(X) = LU$  be a Levi decomposition. Then there exists an  $L$ -stable subvariety  $\bar{X}_{\text{el}} \subseteq X$  (everything defined over  $k$ ) such that:*

- i) the action of  $L$  on  $\bar{X}_{\text{el}}$  is elementary;*
- ii) the natural morphism  $Q_k(X) \times^L \bar{X}_{\text{el}} \rightarrow X$  is an open embedding;*
- iii) the boundary  $\partial X_0 = X \setminus X_0$  of the open set  $X_0 := U \cdot \bar{X}_{\text{el}} \subseteq X$  is of pure codimension one in  $X$  and none of its irreducible components is  $G$ -stable.*

*Proof.* Let  $X_0 \subseteq X$  be the largest open subset on which the rational map  $u : X \dashrightarrow U$  is regular. Put  $\bar{X}_{\text{el}} := u^{-1}(1) \subseteq X_0$ . Since  $u$  is  $U$ -equivariant, we infer that  $U \times \bar{X}_{\text{el}} \rightarrow X_0$  is an isomorphism which shows *ii*). Property *i*) follows from the fact that  $X_{\text{el}}$  is open in  $\bar{X}_{\text{el}}$ .

Thus, the main point is to show *iii*). Since  $U$  is affine, a point  $x \in X$  is in  $X_0$  if and only if  $u^*(h)$  is regular in a neighborhood of  $x$  for all  $h \in K[U]$ . In a normal variety this is the case if and only if  $x$  is not contained in the pole divisor of any function  $u^*(h)$ . Thus,  $\partial X_0$  is the union of all these pole divisors and therefore is of pure codimension 1.

Finally, suppose  $\partial X_0$  has an irreducible component  $D$  which is  $G$ -stable (but possibly not defined over  $k$ ). Since  $v_D$  is a  $G$ -invariant valuation, Lemma 6.1 implies  $v_D(u^*(h)) \geq 0$  for all  $h \in K[U]$ . Thus, the generic point of  $D$  lies in  $X_0$  in contradiction to  $D \subseteq \partial X_0$ .  $\square$

From Proposition 4.5 we get:

**6.3. Corollary.** *Let  $X$  be a  $k$ -dense  $G$ -variety and let  $D \subseteq X$  be a  $k$ -dense irreducible divisor. Then  $Q_k(D) = Q_k(X)$ .*

Next, we want to relate  $k$ -central valuations with toroidal embeddings. For this, fix a  $k$ -central valuation  $v$ . It induces an element  $a := \lambda_v \in \mathcal{N}_k(X) = \mathcal{N}(A_k)$  (with  $A_k := A_k(X)$ ) and let  $\mathcal{R}_v := \mathbb{Q}_{\geq 0}a \subseteq \mathcal{N}(A_k) = \mathcal{N}_k(X)$  be the ray generated by  $a$ . Then the theory of toroidal embeddings asserts the existence of an embedding  $\overline{A} = A_k \cup D(a)$  where  $D(a)$  is a homogeneous divisor which is characterized by the fact that  $\lambda_{v_{D(a)}} \sim a$ .

Another way to relate  $\overline{A}$  with  $a$  goes as follows: The space  $\mathcal{N}(A_k)$  contains the group  $\text{Hom}(\mathbf{G}_m, A_k)$  of cocharacters as a sublattice. Thus, there is a homomorphism  $\overline{a} : \mathbf{G}_m \rightarrow A_k$  with  $\overline{a} = ca$  for some  $c \in \mathbb{Q}_{>0}$ . Then

$$(6.5) \quad \pi_a : \overline{A} \rightarrow \overline{A} : x \mapsto \lim_{t \rightarrow 0} \overline{a}(t)x$$

is a well-defined projection onto  $D(a)$ .

There is also a slightly more general version of this construction. For this, let  $X \rightarrow Y$  be a principal  $A_k$ -bundle (defined over  $k$ ) and let  $X(a) := X \times^{A_k} \overline{A} \rightarrow Y$  be the associated fiber bundle. Then  $X$  is open in  $X(a)$  and the complement is an irreducible divisor  $D_a$ . Let  $v_0 = v_{D_a}$  be the associated valuation. Since  $K(X)^{A_k} = K(Y)$ , the valuation  $v_0$  is  $k$ -central for the action of  $A_k$  with  $\lambda_{v_0} = a$ . This shows, in particular, that  $\mathcal{Z}_k(X) = \mathcal{N}_k(X)$  and therefore, that every  $k$ -central valuation is of this type. Clearly, (6.5) still defines a projection  $\pi_a : X(a) \rightarrow D_a$ .

Now we treat the general case:

**6.4. Lemma.** *Let  $X$  be a normal  $k$ -dense  $G$ -variety and let  $D \subseteq X$  be an irreducible  $G$ -stable divisor such that  $v_D$  is  $k$ -central. Put  $a = \lambda_{v_D}$ . Then there exist a slice  $X_{\text{el}} \subseteq \overline{X}_{\text{el}}$  as in Theorem 6.2 with  $X_{\text{el}}(a) \subseteq \overline{X}_{\text{el}}$  open and  $D_a \subseteq D \cap \overline{X}_{\text{el}}$ .*

*Proof.* Let  $D_{\text{el}} := D \cap \overline{X}_{\text{el}}$ . We have to show that the valuation  $v_{D_{\text{el}}}$  of  $X_{\text{el}}$  equals the valuation  $v_{D_a}$ . For any  $f \in K(X_{\text{el}})$  let  $u^*f \in K(X)$  be its pull-back via the rational map  $u : X \dashrightarrow X_{\text{el}}$ . Then

$$(6.6) \quad v_D(u^*f) = v_{D_{\text{el}}}(f)$$

by Theorem 6.2. Let  $f \in K(X_{\text{el}})^A$ . Then  $u^*f \in K(X)^{AN}$  and therefore  $v_{D_{\text{el}}}(f) = 0$  since  $v_D$  is  $k$ -central. This means that  $v_{D_{\text{el}}}$  is  $k$ -central for the  $A$ -action on  $X_{\text{el}}$ . Because of the injectivity of the map (5.4) it suffices to check  $v_{D_{\text{el}}}(f) = v_a(f)$  for  $A$ -semiinvariants. But this is clear:

$$(6.7) \quad v_{D_a}(f_\chi) = a(\chi) = \lambda_{v_D}(\chi) = v_D(u^*f_\chi) = v_{D_{\text{el}}}(f_\chi). \quad \square$$

Now we introduce another important property of invariant valuations.

**6.5. Definition.** Let  $X$  be a  $k$ -dense variety. A geometric valuation  $v$  of  $X$  is called  *$k$ -dense* if either it is trivial or the corresponding divisor  $D$  of any  $k$ -model of  $v$  is defined over  $k$  and is  $k$ -dense. In this case, we define the complexity of  $v$  as  $c_k(v) = c_k(D)$ .

The next theorem characterizes  $k$ -central valuations. It implies in particular that all  $k$ -central valuations are  $k$ -dense.

**6.6. Theorem.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then for a  $G$ -invariant valuation  $v$  of  $K(X)$  the following conditions are equivalent:*

- i)  $v$  is  $k$ -central
- ii)  $v$  is  $k$ -dense and the restriction of  $v$  to  $K(X)^P = K(X_{\text{an}})^M$  is trivial.

iii)  $v$  is  $k$ -dense with  $c_k(v) = c_k(X)$ .

iv)  $v$  is  $k$ -dense and  $K$ -central.

*Proof.* If  $v$  is trivial then all conditions hold. So assume  $v \neq 0$ . We claim that all conditions imply that  $v$  is geometric. For *i*) this follows from [Kno93, Satz 4.4] and the fact that  $v$  is  $K$ -central. For *ii*) and *iii*) this is part of the definition of  $k$ -density. Thus, without loss of generality we may assume that  $X$  is normal and that it contains a divisor  $D$  with  $v = v_D$ .

*i*) $\Rightarrow$ *ii*). From  $AN \subseteq P$  we get  $K(X)^P \subseteq K(X)^{AN}$ . Thus,  $v$  is trivial on  $K(X)^P$ . Now consider the situation of Lemma 6.4. Since  $X$  is  $k$ -dense also  $X_{\text{el}}$  is  $k$ -dense. Then, because of the projection  $\pi_a : X_{\text{el}} \rightarrow D(a)$  (see (6.5)), also  $D(a)$  is  $k$ -dense. Therefore  $D$  which is birational to  $U \times D(a)$  is  $k$ -dense, as well.

*ii*) $\Rightarrow$ *i*). We first treat the case that the action of  $G$  on  $X$  is elementary. Then clearly we may assume that  $G = MA$  is elementary itself. In that case, we have  $P = G$  and  $N = 1$ . Then using Corollary 4.4 we may assume without loss of generality that  $X$  and  $D$  are affine and smooth.

Now consider the categorical quotient  $\pi : X \rightarrow Y := X//A$ . Then  $Z := \pi(D) = D//A$  is a closed subset of  $Y$ . If  $Z \subsetneq Y$  there is an  $M$ -invariant function  $f_0 \neq 0$  on  $Y$  which vanishes on  $Z$  (Proposition 3.13). Its pull-back  $f = \pi^*(f_0)$  would be a  $G$ -invariant function on  $X$  with  $v(f) > 0$  in contradiction to the assumption.

Thus  $\pi : D \rightarrow Y$  is surjective. This already shows that  $v(f) = 0$  for all non-zero  $f \in K[Y] = K[X]^A$ . It remains to show that  $K(X)^A$  is the quotient field of  $K[X]^A$ . This is equivalent to the generic fiber  $X_y = \pi^{-1}(y)$  of  $\pi$  containing a dense open  $A$ -orbit. Since  $X_y$  is smooth and irreducible it suffices to show that  $X_y$  contains an orbit of dimension  $r = \dim X - \dim Y$ . Recall that  $X_y$ , as a fiber of a categorical quotient, contains exactly one closed  $A$ -orbit. By construction, all  $G$ -orbits of  $D$  are closed. This implies that  $X_y \cap D$  is the closed  $A$ -orbit in  $X_y$ . Since  $y$  is generic, its codimension is 1. Moreover, it is in the closure of every  $A$ -orbit in  $X_y$ . This shows that  $X_y$  contains an orbit of dimension  $r$  proving our claim.

Now let  $X$  be any  $G$ -variety. Then Theorem 6.2 yields an  $L$ -stable elementary slice  $\overline{X}_{\text{el}} \subseteq X$  such that  $D_{\text{el}} := \overline{X}_{\text{el}} \cap D \neq \emptyset$ . Put  $v_{\text{el}} := v_{D_{\text{el}}}$ . Since  $D$  is birational to  $U \times D_{\text{el}}$  also  $v_{\text{el}}$  is  $k$ -dense. Because of  $K(X)^P = K(X_{\text{el}})^{MA}$ ,  $v_{\text{el}}$  meets both conditions of *ii*). Thus, by the above,  $v_{\text{el}}$  is  $k$ -central. Then  $K(X)^{AN} = K(X_{\text{el}})^A$  shows that  $v$  is  $k$ -central.

The equivalence of *ii*) and *iii*) follows from Proposition 4.13. The equivalence of *iv*) with the other conditions follows from  $K(X)^P \subseteq K(X)^B \subseteq K(X)^{AN}$ .  $\square$

**6.7. Remark.** One can show that the complexity of a non-central  $k$ -dense valuation equals  $c_k(X) - 1$ .

For a  $k$ -spherical variety  $X$  we have  $K(X)^P = K$ . Hence:

**6.8. Corollary.** *Let  $X$  be a  $k$ -spherical  $G$ -variety. Then a  $G$ -invariant valuation on  $X$  is  $k$ -central if and only if it is  $k$ -dense.*

We conclude this section with a statement saying that subvarieties of maximal  $k$ -complexity are in a sense controlled by  $k$ -central valuations. Recall that a subvariety  $Y$  of  $X$  is the center of a valuation  $v$  if  $v \geq 0$  on the local ring  $\mathcal{O}_{X,Y}$  and  $v > 0$  on its maximal ideal



$\mathfrak{m}_{X,Y}$ . If  $v$  is geometric with model  $(\tilde{X}, D)$  this means that the birational map  $\tilde{X} \dashrightarrow X$  maps  $D$  dominantly to  $Y$ . The center is unique if it exists. More generally, let  $\tilde{X} \rightarrow X$  be just a dominant morphism and  $v$  a valuation of  $\tilde{X}$ . Then  $Y \subseteq X$  is the center of  $v$  if it is the center of the restricted valuation  $v|_{K(X)}$ .

**6.9. Lemma.** *Let  $\pi : \tilde{X} \rightarrow X$  be a dominant  $G$ -morphism where  $\tilde{X}$  is  $k$ -dense and  $X$  is locally linear. Let  $Y \subseteq X$  be a  $k$ -dense closed  $G$ -subvariety with  $c_k(Y) = c_k(\tilde{X})$ . Then  $Y$  is the center of a  $k$ -central valuation of  $\tilde{X}$ .*

*Proof.* The strategy is to construct a dominant rational  $G$ -map  $\Phi : Z \times \mathbf{A}^1 \dashrightarrow \tilde{X}$  where  $Z$  is a  $k$ -dense  $G$ -variety,  $\pi \circ \Phi$  is defined in  $D_0 := Z \times 0$ , and  $\pi \circ \Phi$  maps  $D_0$  dominantly to  $Y$ . In that case,  $\Phi$  induces an embedding  $k(\tilde{X}) \hookrightarrow k(Z \times \mathbf{A}^1)$  and we claim that we can take for  $v$  the restriction of  $v_{D_0}$  to  $k(\tilde{X})$ .

First,  $v$  is geometric and  $G$ -invariant since  $v_{D_0}$  is. Let  $(\bar{X}, D)$  be a model of  $v$ . Then the rational map  $Z \times \mathbf{A}^1 \dashrightarrow \bar{X}$  maps  $D_0$  dominantly to  $D$ . This implies that  $v$  is  $k$ -dense since  $Z$  is  $k$ -dense. Furthermore, since  $\pi \circ \Phi$  maps  $D_0$  dominantly to  $Y$ , the divisor  $D$  is mapped by  $\pi$  dominantly to  $Y$ , i.e.,  $Y$  is the center of  $v$  in  $X$ . Finally, since  $c_k(v) \geq c_k(Y) = c_k(\tilde{X})$ , the valuation  $v$  is  $k$ -central by Theorem 6.6.

To construct  $\Phi$ , we first apply the Local Structure Theorem 4.2 to  $Y \subseteq X$ . With the notation of that theorem, we have  $\dim Y_{\text{el}}//L = \text{trdeg } k(Y)^P = c_k(Y)$ . On the other hand, we have  $\dim R//L \leq c_k(X) \leq c_k(\tilde{X})$ . Since  $c_k(Y) = c_k(\tilde{X})$  and since  $Y_{\text{el}}//L \rightarrow R//L$  is a closed embedding, we see that actually  $Y_{\text{el}}//L = R//L =: S$ . This means that every  $L$ -orbit of  $R$  contains a unique  $L$ -orbit of  $Y_{\text{el}}$  in its closure.

Let  $\tilde{R} := \pi^{-1}(R) \subseteq \tilde{X}$  such that  $U \times \tilde{R} \rightarrow \tilde{X}$  is birational. Now we would like to apply Kempf's Theorem 3.6 simultaneously to all points of the form  $\pi(x)$  with  $x \in \tilde{R}$ . To this end, consider the field extension  $F := k(\tilde{R})$  of  $k(S)$  and the affine  $F$ -variety  $R_F := R \times_S \text{Spec } F$  obtained by base change. Then  $R_F$  contains just one closed  $L_F$ -orbit namely  $Y_F := Y_{\text{el}} \times_S \text{Spec } F$ . The generic point  $\eta = \text{Spec } F$  of  $\tilde{R}$  can be considered as an  $F$ -rational point of  $R_F$ . Now we apply Kempf's theorem to  $L_F$  acting on  $R_F$ , the point  $\eta$ , and the closed subset  $Y_F$ . Then this yields a 1-parameter subgroup  $\lambda : \mathbf{G}_{m,F} \rightarrow L_F$  (defined over  $F$ ) such that the orbit map  $\mathbf{G}_{m,F} \rightarrow R_F : t \mapsto \lambda(t)\eta$  extends to a morphism  $\Phi' : \mathbf{A}_F^1 \rightarrow R_F$  with  $\Phi'(0) \in Y_F$ .

Unraveling the definitions, the morphism  $\lambda$  corresponds to a rational morphism  $\tilde{R} \times \mathbf{G}_m \dashrightarrow L : (x, t) \mapsto \lambda_x(t)$ . The orbit map  $\Phi_0 : \tilde{R} \times \mathbf{A}^1 \dashrightarrow \tilde{R} : (x, t) \mapsto \lambda_x(t)x$  has the property that  $\Phi' = \pi \circ \Phi_0 : \tilde{R} \times \mathbf{A}^1 \rightarrow R : (x, t) \mapsto \lambda_x(t)\pi(x)$  is defined in  $t = 0$  with  $Y_0 := \Phi'(\tilde{R} \times 0) \subseteq Y_{\text{el}}$ . The fact that  $Y_F$  is an  $L_F$ -orbit means that  $Y_0$  meets the generic  $L$ -orbit of  $Y_{\text{el}}$ . Moreover  $\Phi_0$  is dominant because of  $\Phi_0(x, 1) = x$  for all  $x \in \tilde{R}$ .

Finally, we extend  $\Phi_0$  to  $\tilde{X}$  by using the birational map  $U \times \tilde{R} \xrightarrow{\sim} \tilde{X}$ . For this we define  $Z := G \times \tilde{X}$  and  $\Phi$  as the composition of

$$(6.8) \quad G \times \tilde{X} \times \mathbf{A}^1 \xrightarrow{\sim} G \times U \times \tilde{R} \times \mathbf{A}^1 \xrightarrow{\Phi_0} G \times U \times \tilde{R} \xrightarrow{\sim} G \times X \rightarrow X : (g, ux, t) \mapsto gu\lambda(x, t)x.$$

Then  $\Phi$  is clearly equivariant and dominant. Moreover,  $D_0$  is mapped dominantly to  $Y$  because  $GY_0 \supseteq ULY_0$  is dense in  $Y$ .  $\square$

A first application is:

**6.10. Corollary.** *Let  $\varphi : X \rightarrow Y$  be a dominant  $G$ -equivariant morphism between  $k$ -dense  $G$ -varieties. Then  $\varphi$  induces a surjective map  $\varphi_* : \mathcal{N}_k(X) \rightarrow \mathcal{N}_k(Y)$  such that  $\varphi_*\mathcal{Z}_k(X) \subseteq \mathcal{Z}_k(Y)$  with equality if  $c_k(X) = c_k(Y)$ . In particular, if  $c_k(X) = c_k(Y)$  and  $\mathrm{rk}_k X = \mathrm{rk}_k Y$  then  $\varphi_* : \mathcal{Z}_k(X) \rightarrow \mathcal{Z}_k(Y)$  is bijective.*

*Proof.* The morphism  $\varphi$  induces obviously an injective homomorphism  $\Xi_k(Y) \hookrightarrow \Xi_k(X)$  of which  $\varphi_*$  is the dual, hence surjective map. Because of  $\varphi^*K(Y)^{AN} \subseteq K(X)^{AN}$ , the restriction of a central valuation on  $K(X)$  to  $K(Y)$  is central, hence  $\varphi_*\mathcal{Z}_k(X) \subseteq \mathcal{Z}_k(Y)$ .

Now assume  $c_k(X) = c_k(Y)$  and let  $v \in \mathcal{Z}_k(Y)$ . Let  $(\bar{Y}, D)$  be a model of  $v$ . Since  $c_k(D) = c_k(Y) = c_k(X)$ , Lemma 6.9 implies that  $D$  is the center of some  $\bar{v} \in \mathcal{Z}_k(X)$ . This proves  $\varphi_*\mathcal{Z}_k(X) \subseteq \mathcal{Z}_k(Y)$ . The last assertion is clear.  $\square$

**6.11. Remark.** If  $c_k(Y) < c_k(X)$  then  $\varphi_*$  is in general not surjective on  $k$ -central valuations. Let, for example,  $k = \mathbb{C}$ ,  $G = SL(2)$  and  $X = SL(2)$  with  $G$  acting on the left. Then  $\mathcal{Z}_k(X) = \mathbb{Q}_{>0}$  and  $c_k(X) = 1$  (easy, see e.g. [LV83]). Now let  $Y = \mathbb{C}^2 \setminus \{0\} = G/N$ . Then  $\mathcal{Z}_k(Y) = \mathbb{Q}$  and  $c_k(Y) = 0$  (see section §8). In this case, the surjective morphism  $X \rightarrow Y$  is not surjective on central valuations.

## 7. Toroidal embeddings

We have seen in Lemma 6.4 that the neighborhood of a  $k$ -central divisor is modeled after a toroidal embedding corresponding to a ray, i.e., a one-dimensional cone. In this section we generalize this construction to other toroidal embeddings.

Recall that the theory of toroidal embedding (see [KKMSD73] or [Oda88]) attaches to every fan  $\mathcal{F}$  in  $\mathcal{N}_k := \mathcal{N}_k(X)$  a normal equivariant embedding  $A_k \hookrightarrow A(\mathcal{F})$ . Recall that a fan is a finite collection of finitely generated strictly convex cones  $\mathcal{C} \subseteq \mathcal{N}_k$  satisfying certain axioms. The embedding  $A(\mathcal{F})$  has the property that to each  $\mathcal{C} \in \mathcal{F}$  there corresponds an orbit  $A(\mathcal{C}) \subseteq A(\mathcal{F})$ . Its closure  $\overline{A(\mathcal{C})} = \overline{A(\mathcal{C})}$  is

$$(7.1) \quad \overline{A(\mathcal{C})} = \bigcup_{\mathcal{C}' \in \mathcal{F}} A(\mathcal{C}')$$

The *support*  $\mathrm{supp} \mathcal{F}$  of  $\mathcal{F}$  is the union of all  $\mathcal{C} \in \mathcal{F}$ .

More generally, if  $X \rightarrow Y$  is a principal  $A_k$ -bundle then we denote by  $X(\mathcal{F}) := X \times^{A_k} A(\mathcal{F})$  the corresponding relative toroidal embedding over  $Y$ .

Now we generalize this to arbitrary  $k$ -dense  $G$ -varieties:

**7.1. Theorem.** *Let  $X$  be a normal  $k$ -dense  $G$ -variety with LST-slice  $X_{\mathrm{el}}$  and assume that  $X = G \cdot X_{\mathrm{el}}$ . Let  $\mathcal{F}$  be a fan with  $\mathrm{supp} \mathcal{F} \subseteq \mathcal{Z}_k(X)$ . Then there is a unique equivariant embedding  $X \hookrightarrow X(\mathcal{F})$  such that the open embedding  $U \times X_{\mathrm{el}} \rightarrow X$  extends to an open embedding  $U \times X_{\mathrm{el}}(\mathcal{F}) \rightarrow X(\mathcal{F})$  with  $X(\mathcal{F}) = G \cdot X_{\mathrm{el}}(\mathcal{F})$ .*

*Proof.* Let  $X_0$  be any normal, but not necessarily equivariant model of  $X$ . Since each  $\xi \in \mathfrak{g}$  acts as a derivation on  $K(X)$  it induces a rational vector field  $\xi_*$  on  $X_0$ . Luna-Vust have shown ([LV83, Prop. 1.4]) that there is a normal  $G$ -variety  $\overline{X}$  containing  $X_0$  as an open subset if and only if for every  $\xi \in \mathfrak{g}$  the vector field  $\xi_*$  is regular on  $X_0$ . The condition  $\overline{X} = G \cdot X_0$  makes this model even unique.

We claim that the Luna-Vust condition is satisfied for  $X_0 = U \times X_{\mathrm{el}}(\mathcal{F})$ . Since  $X_0$  is normal it suffices to show that  $\xi_*$  is regular in codimension one. Regularity holds

certainly on  $U \times X_{\text{el}}$  since that set is open in the  $G$ -variety  $X$ . The complement consists of divisors  $D$  which correspond to the one-dimensional cones (i.e. rays)  $\mathcal{R} \in \mathcal{F}$ . Because, by assumption,  $\mathcal{R} \subseteq \mathcal{Z}_k(X)$ , the set  $(U \times X_{\text{el}}) \cup D$  is isomorphic to an open subset of a  $G$ -variety by Lemma 6.4. Thus,  $\xi_*$  is regular along  $D$ , as well, proving the claim.  $\square$

**7.2. Remark.** The condition  $X = G \cdot X_{\text{el}}$  is very mild. For example it is clearly satisfied when  $G$  acts transitively on  $X$ . In general, let  $X' := G \cdot X_{\text{el}} = G \cdot (U \cdot X_{\text{el}})$ . Then  $X'$  is a  $G$ -stable open subset to which the theorem applies.

We now study the orbit structure of  $X(\mathcal{F})$ . For every  $\mathcal{C} \in \mathcal{F}$  let

$$(7.2) \quad X_{\text{el}}(\mathcal{C}) := X_{\text{el}} \times^{A_k} A(\mathcal{C}) \text{ and } \overline{X}_{\text{el}}(\mathcal{C}) := \overline{X}_{\text{el}} \times^{A_k} \overline{A}(\mathcal{C})$$

be the relative  $A_k$ -orbit corresponding to  $\mathcal{C}$  and its closure, respectively. The sets  $X_{\text{el}}(\mathcal{C})$  form a stratification of  $X_{\text{el}}(\mathcal{F})$  which extends to  $X(\mathcal{F})$ :

**7.3. Theorem.** *Let  $X \subseteq X(\mathcal{F})$  is in Theorem 7.1. For each  $\mathcal{C} \in \mathcal{F}$  put*

$$(7.3) \quad X(\mathcal{C}) := G \cdot X_{\text{el}}(\mathcal{C}) \text{ and } \overline{X}(\mathcal{C}) := G \cdot \overline{X}_{\text{el}}(\mathcal{C}).$$

*i) The  $X(\mathcal{C})$  form a stratification of  $X(\mathcal{F})$ . More precisely, each  $X(\mathcal{C})$  is locally closed, the closure of  $X(\mathcal{C})$  is  $\overline{X}(\mathcal{C})$ , and  $\overline{X}(\mathcal{C})$  is the disjoint union of all  $X(\mathcal{C}')$  with  $\mathcal{C} \subseteq \mathcal{C}' \in \mathcal{F}$ .*

*ii) The stratification of  $X(\mathcal{F})$  is compatible with that of  $X_{\text{el}}(\mathcal{F})$ , i.e.,  $X(\mathcal{C}) \cap X_{\text{el}}(\mathcal{F}) = X_{\text{el}}(\mathcal{C})$  and  $\overline{X}(\mathcal{C}) \cap X_{\text{el}}(\mathcal{F}) = \overline{X}_{\text{el}}(\mathcal{C})$  for all  $\mathcal{C} \in \mathcal{F}$ . Moreover,  $U \times X_{\text{el}}(\mathcal{C}) \rightarrow X(\mathcal{C})$  and  $U \times \overline{X}_{\text{el}}(\mathcal{C}) \rightarrow \overline{X}(\mathcal{C})$  are open embeddings.*

*iii) Each stratum  $X(\mathcal{C})$  is  $k$ -dense with*

$$\begin{aligned} \dim X(\mathcal{C}) &= \dim X - \dim \mathcal{C}, \quad c_k(X(\mathcal{C})) = c_k(X), \\ \text{rk}_k X(\mathcal{C}) &= \text{rk}_k X - \dim \mathcal{C} \\ Q_k(X(\mathcal{C})) &= Q_k(X), \quad X(\mathcal{C})_{\text{el}} = X_{\text{el}}(\mathcal{C}), \quad X(\mathcal{C})_{\text{an}} = X_{\text{an}} \end{aligned}$$

*iv) Let  $\langle \mathcal{C} \rangle \subseteq \mathcal{N}_k(X)$  be the subspace spanned by  $\mathcal{C}$ . Then*

$$\overline{\Xi}_k(X(\mathcal{C})) = \overline{\Xi}_k(X) \cap \langle \mathcal{C} \rangle^\perp, \quad \mathcal{N}_k(X(\mathcal{C})) = \mathcal{N}_k(X) / \langle \mathcal{C} \rangle$$

*and  $\mathcal{Z}_k(X(\mathcal{C}))$  is the image of  $\mathcal{Z}_k(X)$ .*

*Proof.* We start with a general remark: because of  $X(\mathcal{F}) = G \cdot X_{\text{el}}(\mathcal{F})$  we have

$$(7.4) \quad Z = G \cdot (Z \cap X_{\text{el}}(\mathcal{F})) \text{ for all } G\text{-stable subsets } Z \subseteq X(\mathcal{F}).$$

Let  $\tilde{X}(\mathcal{C})$  be closure of  $U \times \overline{X}_{\text{el}}(\mathcal{C})$  in  $X(\mathcal{F})$ . We claim that  $\tilde{X}(\mathcal{C}) = \overline{X}(\mathcal{C})$ . For this, we consider the case when  $\dim \mathcal{C} = 1$  first. Let  $a \in \mathcal{C}$  be a generator, let  $v \in \mathcal{Z}_k(X)$  be the corresponding  $k$ -central valuation, and let  $(\overline{X}, D)$  be a model of  $v$ . Then it follows from Lemma 6.4 that the birational map  $\overline{X} \dashrightarrow X(\mathcal{F})$  is regular in  $D$  and maps  $D$  to  $U \times X_{\text{el}}(\mathcal{C})$ . Because  $D$  is  $G$ -stable, this implies that also  $\tilde{X}(\mathcal{C})$  is  $G$ -stable. By construction, we have  $\tilde{X}(\mathcal{C}) \cap X_{\text{el}}(\mathcal{F}) = \overline{X}_{\text{el}}(\mathcal{C})$ . Thus (7.4) with  $Z = \tilde{X}(\mathcal{C})$  yields  $\tilde{X}(\mathcal{C}) = \overline{X}(\mathcal{C})$ .

Next let  $\mathcal{C} \in \mathcal{F}$  be arbitrary, let  $\mathcal{R}_1, \dots, \mathcal{R}_s \subseteq \mathcal{C}$  be its extremal rays, and put

$$(7.5) \quad X'(\mathcal{C}) := \overline{X}(\mathcal{R}_1) \cap \dots \cap \overline{X}(\mathcal{R}_s).$$

which is a  $G$ -stable subset of  $X(\mathcal{F})$ . Then

$$(7.6) \quad X'(\mathcal{C}) \cap (U \times X_{\text{el}}(\mathcal{F})) = \bigcap_{i=1}^s (U \times \overline{X}_{\text{el}}(\mathcal{R}_i)) = U \times \overline{X}_{\text{el}}(\mathcal{C}) = \tilde{X}(\mathcal{C}) \cap (U \times \overline{X}_{\text{el}}(\mathcal{F})).$$

This implies that  $\tilde{X}(\mathcal{C})$  is an irreducible component of  $X'(\mathcal{C})$  and therefore  $G$ -stable, as well. From (7.4) with  $Z = \tilde{X}(\mathcal{C})$  we conclude  $\tilde{X}(\mathcal{C}) = \overline{X}(\mathcal{C})$ , proving the claim.

We have proved that  $\overline{X}(\mathcal{C})$  is closed with

$$(7.7) \quad \overline{X}(\mathcal{C}) \cap (U \times X(\mathcal{F})) = U \times \overline{X}_{\text{el}}(\mathcal{C}).$$

Next we define the  $G$ -stable subvariety

$$(7.8) \quad X^0(\mathcal{C}) := \overline{X}(\mathcal{C}) \setminus \bigcup_{\mathcal{C}' \not\subseteq \mathcal{C}} \overline{X}(\mathcal{C}')$$

Then (7.7) implies  $X^0(\mathcal{C}) \cap X(\mathcal{F}) = X(\mathcal{C})$  and therefore  $X^0(\mathcal{C}) = X(\mathcal{C})$  by (7.4). This shows *ii*) and that  $X(\mathcal{C})$  is an open subset of  $\overline{X}(\mathcal{C})$ . The rest of *i*) follows easily with the help of (7.1). All assertions in *iii*) follow from *ii*) and the corresponding assertions for  $X_{\text{el}}(\mathcal{C})$ . The same holds for all assertions in *iv*) except for the final one. That follows, e.g., from the statement over  $K$  which is proved in [Kno93, Satz 7.4.5].  $\square$

**7.4. Remarks.** *i*) By shrinking  $X_{\text{an}}$  and thereby  $X(\mathcal{F})$ , we can achieve that any finite number of “generic” conditions hold for all strata. For example: all strata  $X(\mathcal{C})$  admit an orbit space  $X(\mathcal{C})/G$ ; stratum, orbit space and quotient map are smooth; all isotropy groups in a stratum have  $K$ -isomorphic Levi complements etc.

*ii*) In general,  $X(\mathcal{F})$  is only defined if  $X$  meets the conditions of Theorem 7.1. So, when we speak about  $X(\mathcal{F})$  we mean  $X_0(\mathcal{F})$  where  $X_0 = G \cdot X_{\text{el}}$  is an open subset depending on the choice of  $X_{\text{el}}$ . In this sense,  $X(\mathcal{F})$  is only unique up to a birational map which is defined in all strata. Observe that in the most important case, namely when  $X$  is homogeneous and  $k$ -spherical,  $X(\mathcal{F})$  is unambiguously defined embedding of  $X$  since then all strata are orbits.

*iii*) The dimension of the generic  $G$ -orbits in  $X(\mathcal{C})$  is not so easy to control. An upper bound for the codimension is  $c_k(X)$ . Moreover we show in Section 8 that this bound is attained as soon as  $\mathcal{C}$  contains an interior point of  $\mathcal{Z}_k(X)$ . This is in particular the case when  $\dim \mathcal{C} = \text{rk}_k X$ .

As usual, for  $k$ -spherical varieties we can be more specific:

**7.5. Corollary.** *Let  $X$  be a homogeneous  $k$ -spherical variety and let  $X \hookrightarrow X(\mathcal{F})$  be the embedding corresponding to a fan  $\mathcal{F}$  with  $\text{supp } \mathcal{F} \subseteq \mathcal{Z}_k(X)$ . Then the strata  $X(\mathcal{C})$  are precisely the  $G$ -orbits. In particular, all  $G$ -orbits in  $X(\mathcal{F})$  are  $k$ -dense.*

*Proof.* The group  $MA$  acts transitively on  $X_{\text{an}}$  and therefore on  $X_{\text{el}}(\mathcal{C})$ . Thus,  $G$  acts transitively on  $X(\mathcal{C}) = G \cdot X_{\text{el}}(\mathcal{C})$ .  $\square$

For local fields, the set  $X(k)$  is also stratified:

**7.6. Proposition.** *Assume that  $k$  be a local field. Then the sets  $X(\mathcal{C})(k)$  with  $\mathcal{C} \in \mathcal{F}$  form a stratification of  $X(\mathcal{F})(k)$  for the Hausdorff topology.*

*Proof.* Clearly, each set  $X(\mathcal{C})(k)$  is locally closed and its closure is contained in the union of all sets  $X(\mathcal{C}')(k)$  with  $\mathcal{C} \subseteq \mathcal{C}' \in \mathcal{F}$ . It remains to be shown that conversely each such set  $X(\mathcal{C}')(k)$  is contained in the closure of  $X(\mathcal{C})(k)$ . Now Hilbert's Theorem 90 implies that the fibration  $X_{\text{el}} \rightarrow X_{\text{an}}$  is Zariski locally trivial by . Thus, the construction of  $X(\mathcal{F})$  shows that it is covered by open subsets of the form  $Y \times A(\mathcal{F})$  where  $Y$  is some  $k$ -variety.

Toroidal theory shows, that the choice of a 1-parameter subgroup  $a : \mathbf{G}_m \rightarrow A_k(X)$  in the relative interior of  $\mathcal{C}'$  yields a surjective morphism

$$(7.9) \quad \pi : A(\mathcal{C}) \rightarrow A(\mathcal{C}') : x \mapsto \lim_{t \rightarrow 0} a(t)x.$$

Moreover,  $\pi$  is even a homomorphism of split tori whose kernel is connected. Hence it has a section which shows that the induced map  $A(\mathcal{C})(k) \rightarrow A(\mathcal{C}')(k)$  on rational points is surjective as well. This implies the claim.  $\square$

Next we study morphisms between toroidal embeddings.

**7.7. Proposition.** *Let  $X \subseteq X(\mathcal{F})$  is in Theorem 7.1 and let  $\varphi : X \rightarrow Y$  be a dominant equivariant morphism. Then  $\varphi$  extends to a rational map  $X(\mathcal{F}) \dashrightarrow Y$  which is defined in all strata if and only all  $v$  in the relative interior  $\mathcal{C}^\circ$  of any  $\mathcal{C} \in \mathcal{F}$  have the same center in  $Y$ .*

*Proof.* Suppose  $Z \subseteq Y$  is the common center. Let  $f \in \mathcal{O}_{Y,Z}$ . Then  $v(\varphi^*(f)) \geq 0$  for all  $v \in \mathcal{C}^\circ$  which implies that  $\varphi^*\mathcal{O}_{Y,Z} \subseteq \mathcal{O}_{X(\mathcal{F}),\overline{X}(\mathcal{C})}$ . Hence  $\varphi$  regular in  $\overline{X}(\mathcal{C})$ .  $\square$

**7.8. Corollary.** *Let  $X \hookrightarrow (\mathcal{F})$  and  $Y \hookrightarrow Y(\mathcal{F}')$  be two toroidal embeddings and let  $\varphi : X \rightarrow Y$  be a dominant equivariant morphism. Assume  $\varphi_* : \mathcal{N}_k(X) \rightarrow \mathcal{N}_k(Y)$  maps  $\mathcal{F}$  to  $\mathcal{F}'$  (i.e., for all  $\mathcal{C} \in \mathcal{F}$  there is  $\mathcal{C}' \in \mathcal{F}'$  with  $\varphi_*(\mathcal{C}) \subseteq \mathcal{C}'$ ). Then  $\varphi$  extends to a rational map  $X(\mathcal{F}) \dashrightarrow Y(\mathcal{F}')$  such that a stratum  $X(\mathcal{C})$  is mapped into  $Y(\mathcal{C}')$  whenever  $\varphi_*(\mathcal{C}) \subseteq \mathcal{C}'$ .*

We say that the embedding  $\iota_1 : X \hookrightarrow X_1$  dominates the embedding  $\iota_2 : X \hookrightarrow X_2$  if there is a (necessarily unique) morphism  $\varphi : X_1 \rightarrow X_2$  with  $\iota_2 = \varphi \circ \iota_1$ . Clearly, this defines a partial order on the set of all embeddings of  $X$ .

For toroidal embeddings this means the following: Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two fans supported in  $\mathcal{Z}_k(X)$ . Then  $\mathcal{F}_1$  dominates  $\mathcal{F}_2$  if for every  $\mathcal{C}_1 \in \mathcal{F}_1$  there is an  $\mathcal{C}_2 \in \mathcal{F}_2$  with  $\mathcal{C}_1 \subseteq \mathcal{C}_2$ . Then  $X(\mathcal{F}_1)$  dominates  $X(\mathcal{F}_2)$  if and only if  $\mathcal{F}_1$  dominates  $\mathcal{F}_2$ . Observe, that for the dominance order there is always a maximal element, namely the trivial fan corresponding to the trivial embedding  $X \hookrightarrow X$ . On the other side, if  $X$  is  $k$ -convex there is also a minimal element namely the standard embedding.

Next we show that only toroidal embeddings can dominate a toroidal embedding.

**7.9. Proposition.** *Let  $X$  be a homogeneous  $k$ -spherical variety. Let  $X \hookrightarrow X'$  be a normal equivariant embedding dominating the toroidal embedding  $X(\mathcal{F})$ . Then  $X' = X(\mathcal{F}')$  for some fan  $\mathcal{F}'$  dominating  $\mathcal{F}$ .*

*Proof.* Let  $\varphi : X' \rightarrow X(\mathcal{F})$  be the dominating morphism and put  $X'_{\text{el}} := \varphi^{-1}(X_{\text{el}}(\mathcal{F}))$ . Then  $U \times X'_{\text{el}} \rightarrow X'$  is an open embedding and  $X' = G \cdot X'_{\text{el}}$ . Moreover, since  $X_{\text{an}}$  is homogeneous,  $X'_{\text{el}}$  is of the form  $X_{\text{el}} \times_{X_{\text{an}}} \overline{A}$  where  $\overline{A}$  is an embedding of  $A_k(X)$  dominating  $A(\mathcal{F})$ . But then it follows from the theory of torus embeddings that  $\overline{A} = A(\mathcal{F}')$  for some fan dominating  $\mathcal{F}$ . This implies in turn  $X' = X(\mathcal{F}')$ .  $\square$

Of particular importance is the case when the support of  $\mathcal{F}$  is all of  $\mathcal{Z}_k(X)$  in which case we call  $\mathcal{F}$  *complete (for  $X$ )*. It is not true in general that  $X(\mathcal{F})$  is complete in the algebro-geometric sense but still  $X(\mathcal{F})$  has the following completeness property with respect to rational points:

**7.10. Theorem.** *Let  $\mathcal{F}$  be a complete fan, let  $X(\mathcal{F}) \hookrightarrow \overline{X}$  be a locally linear equivariant embedding, and let  $Y \subseteq \overline{X}$  be a  $k$ -dense,  $G$ -stable subvariety with  $c_k(Y) = c_k(X)$ . Then  $Y \cap X(\mathcal{F}) \neq \emptyset$ .*

*Proof.* Lemma 6.9 asserts that  $Y$  is the center of some  $v \in \mathcal{Z}_k(X)$ . Since, by assumption,  $\lambda_v \in \text{supp } \mathcal{F}$  it has a center in  $X(\mathcal{F})$  (namely  $\overline{X}(\mathcal{C})$  with  $v$  in the interior of  $\mathcal{C}$ ). Thus  $Y \cap X(\mathcal{F}) \neq \emptyset$ .  $\square$

For  $k$ -spherical varieties this means that the complement of  $X(\mathcal{F})$  cannot contain  $k$ -rational points at all. Thus we get that  $X(\mathcal{F})(k)$  is maximal in the following sense:

**7.11. Corollary.** *Let  $X$  be a  $k$ -spherical variety and let  $X(\mathcal{F}) \hookrightarrow \overline{X}$  be a locally linear equivariant embedding. Then  $X(\mathcal{F})(k) = \overline{X}(k)$ .*

For local fields we get a true compactification:

**7.12. Corollary.** *Assume that  $k$  is a local field and let  $X$  be a  $k$ -spherical variety. Then  $X(\mathcal{F})(k)$  is compact (with respect to the Hausdorff topology) if and only if  $\mathcal{F}$  is complete.*

*Proof.* First assume  $\mathcal{F}$  to be complete. By [Sum75], there is an equivariant embedding of  $X(\mathcal{F})$  into a normal complete variety  $\overline{X}$ . Then  $X(\mathcal{F})(k) = \overline{X}(k)$  is compact. Conversely, suppose  $\mathcal{F}$  is not complete. Then there is a fan  $\mathcal{F}'$  which properly contains  $\mathcal{F}$ . Because  $X(\mathcal{F})(k)$  is a proper dense (by Proposition 7.6) subset of  $X(\mathcal{F}')$  it cannot be compact.  $\square$

Since a fan consists of strictly convex cones, there is in general no canonical complete fan unless the valuation cone itself is strictly convex (apart from the trivial case when  $\text{rk } X = 1$ ).

**7.13. Definition.** A  $k$ -dense  $G$ -variety is called  *$k$ -convex* if its valuation cone  $\mathcal{Z}_k(X)$  is strictly convex. In that case, we define the *standard fan*  $\mathcal{F}_{\text{st}}$  as the set of faces of  $\mathcal{Z}_k(X)$  and the *standard embedding*  $X_{\text{st}} := X(\mathcal{F}_{\text{st}})$ .

Observe, that Remark 7.4ii) also applies to the standard embedding, i.e.,  $X_{\text{st}}$  is only an embedding and unique in a birational sense unless  $X$  is homogeneous and  $k$ -spherical. We will show in Section 8 that the condition of convexity is not very serious.

By construction,  $X_{\text{st}}$  has a unique closed stratum namely  $X(\mathcal{C})$  for  $\mathcal{C} = \mathcal{Z}_k(X)$ . In particular, if  $X$  is  $k$ -spherical then  $X_{\text{st}}$  is a *simple embedding* meaning that it has a unique closed orbit. This leads to another characterization of the standard embedding.

**7.14. Proposition.** *Let  $X \hookrightarrow X_{\text{st}}$  be the standard embedding of a homogeneous  $k$ -convex,  $k$ -spherical variety  $X$ . Let  $Z$  be a complete locally linear  $G$ -variety which contains exactly one  $k$ -dense orbit of rank zero. Then every  $G$ -equivariant  $k$ -morphism  $X \rightarrow Z$  extends (uniquely) to a morphism  $X_{\text{st}} \rightarrow Z$ .*

*Proof.* Let  $X'$  be the normalization of the closure of the graph of  $\varphi : X \rightarrow Z$  in  $X_{\text{st}} \times Z$ . Then  $X'$  dominates  $X_{\text{st}}$  via the projection  $\pi$  to the first factor. Proposition 7.9 implies

that  $X' = X(\mathcal{F}')$  where  $\mathcal{F}'$  is a fan supported in  $\mathcal{Z}_k := \mathcal{Z}_k(X)$ . The fact that  $Z$  is complete implies that  $\pi$  is proper, hence the support of  $\mathcal{F}'$  is all of  $\mathcal{Z}_k$ . It remains to show that  $\mathcal{F}' = \mathcal{F}_{\text{st}}$  since then  $\pi$  is an isomorphism and  $X_{\text{st}}$  maps to  $Z$  via the second projection.

Now suppose  $\mathcal{F}' \neq \mathcal{F}_{\text{st}}$ . Then  $\mathcal{F}'$  necessarily contains more than one cone of maximal dimension and therefore also a cone  $\mathcal{C}$  of codimension 1 separating two of them. Let  $X_h := \overline{X}(\mathcal{C}) \subseteq X(\mathcal{F}')$ . Then  $X_h$  is of rank 1 by Theorem 7.3 *iv*) and consists of three orbits: the open orbit  $X(\mathcal{C})$  and two closed orbits  $Y_i = X(\mathcal{C}_i)$ ,  $i = 1, 2$ , corresponding to the two cones  $\mathcal{C}_i$  in  $\mathcal{F}'$  having  $\mathcal{C}$  as a face. The morphism  $\pi$  maps  $X_h$  to  $Y := X(\mathcal{Z}_k) \subseteq X_{\text{st}}$ , the closed stratum of  $X_{\text{st}}$ . Thereby the orbits  $Y_1$  and  $Y_2$  are mapped isomorphically to  $Y$ . Moreover, all fibers  $F_y$  of  $X_h \rightarrow Y$  are isomorphic to the projective line  $\mathbf{P}_k^1$ .

Now consider the morphism  $\psi : X_h \hookrightarrow X(\mathcal{F}') \rightarrow Z$ . Then the images of  $Y_1$  and  $Y_2$  under  $\psi$  are both  $k$ -dense orbits of rank zero hence coincide with the unique orbit  $\tilde{Y}$  with  $c_k(\tilde{Y}) = 0$ . According to Corollary 4.4 we can choose a  $P$ -stable affine open subset  $Z^0 \subseteq Z$  with  $Z \cap \tilde{Y} \neq \emptyset$ . Let  $X_h^0 \subseteq X_h$  be its preimage. Then  $Y_i^0 := \pi(Y_i \cap X_h^0)$ ,  $i = 1, 2$ , are non-empty open subsets of  $Y$  thus contain its open  $P$ -orbit  $Y^1$ . We conclude that  $F_y \subseteq X_h^0$  for all  $y \in Y^1$ . But then  $\psi(F_y)$  is a complete irreducible subset of the affine set  $Z^0$ , thus a point. This contradicts the fact that, by construction, the restriction of  $\psi$  to any fiber of  $\pi$  is quasifinite.  $\square$

Following [BT65, 12.3] we call an irreducible representation  $V$  of  $G$  (over  $k$ ) *strongly rational* if it admits a  $P$ -eigenvector. In this case  $\mathbf{P}(V)^{AN}$  consists of just one point. Its orbit is therefore the only one of rank 0.

**7.15. Corollary.** *Let  $X = G/H$  be  $k$ -convex and  $k$ -spherical. Let, moreover,  $V$  be a strongly rational representation of  $G$ . Then every equivariant  $k$ -morphism  $G/H \rightarrow \mathbf{P}(V)$  extends to a morphism  $X_{\text{st}} \rightarrow \mathbf{P}(V)$ .*

## 8. Horospherical varieties

We would like to understand the closed strata of  $X(\mathcal{F})$  when  $\mathcal{F}$  is a complete fan. Since by Theorem 7.3 these strata are  $G$ -varieties of  $k$ -rank 0 we will study rank-0-varieties in general. Over an algebraically closed field, a homogeneous variety of rank zero is known to be a flag variety. Over arbitrary fields, one gets a wider class of spaces which nevertheless share some properties with flag varieties.

To formulate our result, we need to introduce the process of parabolic induction which is also of independent interest.

**8.1. Definition.** Let  $Q = LU \subseteq G$  be a parabolic subgroup containing  $P$  and let  $Q^- = U^-L$  be the opposite parabolic subgroup.

- i)* Let  $Y$  be an  $L$ -variety. Consider it as a  $Q^-$ -variety via the projection  $Q^- \rightarrow L$ . Then  $G \times^{Q^-} Y$  is the *parabolical induction of  $Y$  to  $G$* .
- ii)* Similarly, the parabolical induction of a  $k$ -subgroup  $H \subseteq L$  is the subgroup  $U^-H$  of  $G$ . In that case, the homogeneous variety  $G/U^-H$  is parabolically induced from  $L/H$ .

Most properties are preserved under parabolic induction:

**8.2. Proposition.** *Let  $Q = LU \subseteq G$  be a parabolic subgroup containing  $P$  and  $Q^- = U^-L$  its opposite. Let  $Y$  be a  $k$ -dense  $L$ -variety and let  $X = G \times^{Q^-} Y$  be its parabolical induction. Then:*

- i)  $X$  is  $k$ -dense.*
- ii)  $Q_k(X) = U \cdot Q_k(Y)$ .*
- iii)  $X_{\text{el}} = Y_{\text{el}}$  and  $X_{\text{an}} = Y_{\text{an}}$ .*
- iv)  $c_k^G(X) = c_k^L(Y)$  and  $\text{rk}_k^G(X) = \text{rk}_k^L(Y)$ .*
- v)  $\Xi_k(X) = \Xi_k(Y)$ ,  $\mathcal{N}_k(X) = \mathcal{N}_k(Y)$ , and  $\mathcal{Z}_k(X) = \mathcal{Z}_k(Y)$ .*

*Proof.* The local structure theorem for  $Y$  yields an open embedding  $Q_k(Y)_u \times Y_{\text{el}} \hookrightarrow Y$ . Combined with the open embedding  $U \hookrightarrow G/Q^-$  (the big cell) we get the open embeddings

$$(8.1) \quad U \times Q_k(Y)_u \times Y_{\text{el}} \hookrightarrow U \times Y \hookrightarrow X.$$

Then *i)*, *ii)* and the first part of *iii)* follow immediately from Proposition 4.5. These imply all other assertions except the last one on valuation cones. Because of  $K(X)^U = K(Y)$ , the restriction of any  $k$ -central valuation to  $K(Y)$  is  $k$ -central as well. This shows  $\mathcal{Z}_k(X) \subseteq \mathcal{Z}_k(Y)$ . Now let  $v_0 \in \mathcal{Z}_k(Y)$  and let  $(\overline{Y}, D_0)$  be a model of  $v_0$ . Let  $D := G \times^{Q^-} D_0 \subseteq \overline{X} := G \times^{Q^-} \overline{Y}$ . Then the restriction of  $v_D$  to  $K(Y)$  is equivalent to  $v_0$  which shows the opposite inclusion.  $\square$

We proceed with the classification of varieties of  $k$ -rank 0.

**8.3. Theorem.** *For a  $k$ -dense  $G$ -variety  $X$  the following are equivalent:*

- i)  $\text{rk}_k X = 0$*
- ii) An open dense  $G$ -stable subset of  $X$  is induced from an anisotropic action.*
- iii)  $X = G \cdot X^{AN}$ .*

*Proof.* *i)  $\Rightarrow$  ii)* We apply the Generic Structure Theorem 4.6 to  $X$ . Then  $\text{rk}_k X = 0$  means that  $A$  acts trivially on  $X_{\text{el}}$ , i.e., the action of  $L$  on  $X_{\text{el}}$  is anisotropic. Let  $Q^- = LU^-$  be the opposite parabolic of  $Q$  and let by  $\mathfrak{u}^- = \text{Lie } U^-$ . We claim that  $X_{\text{el}}$  consists of  $U^-$ -fixed points. For this it suffices to show that  $\mathfrak{u}^- x = 0$  where  $x \in X_{\text{el}}$  is any smooth point. Since  $T_x X = \mathfrak{u}x \oplus T_x X_{\text{el}}$ , all weights of  $A$  on  $T_x X$  are either zero or positive restricted roots. On the other hand all weights on  $\mathfrak{u}^- x$  are negative roots. Thus  $\mathfrak{u}^- x = 0$ .

From the claim, we infer that  $X_{\text{el}}$  is a  $Q^-$ -variety with  $U^-$  acting trivially. Now consider the (proper)  $G$ -morphism

$$(8.2) \quad \varphi : \overline{X} := G \times^{Q^-} X_{\text{el}} \rightarrow X : [g, x] \rightarrow gx.$$

By construction, the restriction of  $\varphi$  to the open subset  $U \times X_{\text{el}}$  is an open embedding, hence  $\varphi$  is birational. Let  $X_0 \subseteq \overline{X}$  be the largest open subset on which  $\varphi$  is an open immersion. Since  $X_0$  is  $G$ -stable, it is induced from  $Y = X_0 \cap X_{\text{el}}$  which proves the theorem.

*ii)  $\Rightarrow$  iii)* If  $G \times^{Q^-} Y \subseteq X$  (open) then  $A$  acts trivially on  $Y$ , hence  $Y \subseteq \widetilde{Y} := X^{AN}$ . This implies that  $G \cdot \widetilde{Y}$  is dense in  $X$ . On the other side, the morphism  $G \times^{Q^-} \widetilde{Y} \rightarrow X$  is



proper. Therefore its image  $G \cdot \tilde{Y}$  is also closed, so coincides with  $X$ . From the fact that  $AN^-$  is  $G(k)$ -conjugate to  $AN$  we get  $X = G \cdot X^{AN}$ .

*iii*) $\Rightarrow$ *i*) By assumption, the morphism  $\tilde{X} := G \times^P X^{AN} \rightarrow X$  is surjective. Then  $\text{rk}_k^G \tilde{X} = \text{rk}_k^{MA} X^{AN} = 0$  implies  $\text{rk}_k X = 0$  (Corollary 6.10).  $\square$

**8.4. Remark.** One can show that a locally linear  $X$  has  $k$ -rank zero then there even exist an open subset *containing*  $X(k)$  which is induced from an anisotropic action.

For homogeneous varieties we get:

**8.5. Corollary.** *Let  $H \subseteq G$  be a  $k$ -subgroup. Then the following are equivalent:*

- i)  $\text{rk}_k G/H = 0$ .*
- ii) There is a parabolic subgroup  $Q \subseteq G$  with  $Q_{\text{an}} \subseteq H \subseteq Q$ .*
- iii)  $H$  contains a  $G(k)$ -conjugate of  $AN$ .*
- iv) Let  $\varphi : G/H \hookrightarrow \overline{X}$  be any locally linear  $G$ -equivariant open embedding. Then  $\overline{X}(k) = (G/H)(k)$ .*

*If  $k$  is a local field then these conditions are also equivalent to*

- v)  $(G/H)(k)$  is compact.*
- vi)  $G(k)/H(k)$  is compact.*

*Proof.* *i*) $\Rightarrow$ *ii*) By Theorem 8.3 there is an isomorphism  $G/H \cong G \times^{Q^-} Y$  where  $Y$  is an anisotropic homogeneous  $L$ -variety. From this we get a projection  $\pi : G/H \rightarrow G/Q^-$ . Since  $G(k)$  acts transitively on  $(G/Q^-)(k)$  ([BT65, Thm. 4.13 a)]) we may assume that  $\pi(eH) = eQ^-$ , i.e.,  $H \subseteq Q^-$  and  $Q^-/H \cong Y$ . Thus  $Q_{\text{an}}^- \subseteq H \subseteq Q^-$ .

*ii*) $\Rightarrow$ *iii*) There is  $g \in G(k)$  with  $gPg^{-1} \subseteq Q$  ([BT65, Thm. 4.13 b))). Then  $gANg^{-1} \subseteq Q_{\text{an}} \subseteq H$ .

*iii*) $\Rightarrow$ *i*) Use *iii*) $\Rightarrow$ *i*) from Theorem 8.3.

*i*) $\Rightarrow$ *iv*) Since  $G/H$  is  $k$ -spherical (by *iii*) the assertion follows from Corollary 7.11.

*iv*) $\Rightarrow$ *i*) Suppose  $\text{rk}_k G/H > 0$ . Since then  $\mathcal{Z}_k(G/H) \neq \{0\}$  (by Corollary 5.3) there is a non-trivial  $k$ -central valuation. It belongs to a smooth embedding  $G/H \hookrightarrow X = G/H \cup D$  where  $D$  is  $k$ -dense. Then  $X(k) \not\subseteq (G/H)(k)$ .

The equivalence of *iii*), *v*), and *vi*) is due to Borel-Tits [BT65, Prop. 9.3].  $\square$

Our main application is to toroidal embeddings of  $k$ -spherical varieties:

**8.6. Corollary.** *Assume  $X$  is a homogeneous  $k$ -spherical variety and let  $X \hookrightarrow X(\mathcal{F})$  be the embedding corresponding to a fan  $\mathcal{F}$  with  $\text{supp } \mathcal{F} = \mathcal{Z}_k(X)$ . Then all closed orbits are isomorphic to  $G/H_0$  where  $H_0 = M_0 \cdot Q_k(X)_{\text{an}}^-$  with  $X_{\text{an}} \cong M/M_0$ .*

Our next goal is to understand varieties with  $\mathcal{Z}_k(X) = \mathcal{N}_k(X)$ . For this, we start with the more general problem of studying

$$(8.3) \quad \mathcal{Z}_k^0 = \mathcal{Z}_k(X) \cap (-\mathcal{Z}_k(X)),$$

the largest subspace contained in  $\mathcal{Z}_k(X)$ . The importance of this spaces lies in the fact that  $\mathcal{Z}_k^0$ , when non-zero, prevents  $X$  to have a standard embedding as in Definition 7.13.

**8.7. Definition.** Let  $X$  be a  $k$ -dense  $G$ -variety. An equivariant  $K$ -automorphism of  $K(X)$  is called  $k$ -central if  $K(X)^{(AN)} \subseteq K(X)^{(\varphi)}$ .

Since every  $B$ -semiinvariant is also  $AN$ -semiinvariant, a  $k$ -central automorphism is also  $K$ -central. The latter have been studied in detail in [Kno93, Sec. 8] and [Kno96, Sec. 5]. It was shown that the group of  $K$ -central automorphism is the set of  $K$ -rational points of a subgroup  $\mathfrak{A}_K(X) \subseteq A_K(X)$ . More precisely, a central automorphism  $\varphi$  is related to an element  $a_\varphi \in A_K(X)$  by the formula

$$(8.4) \quad \varphi(f) = \chi_f(a_\varphi)f \quad \text{for all } f \in K(X)^{(B)}.$$

If  $\varphi$  is  $k$ -central then it must act trivially on  $K(X)^{AN}$ . This means that  $\varphi$  is  $k$ -spherical if and only if  $\chi(a_\varphi) = 1$  whenever  $\text{res}_A \chi = 0$ . From this we see that  $k$ -central automorphisms are the points of

$$(8.5) \quad \mathfrak{A}_k(X) = \mathfrak{A}_K(X) \cap A_k(X)$$

which is a  $k$ -subgroup of  $A_k(X)$ .

Central automorphisms are related to central valuations in the following way:

**8.8. Theorem.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then  $\mathcal{N}(\mathfrak{A}_k(X)) = \mathcal{Z}_k^0(X)$ .*

*Proof.* From [Kno93, Satz 8.2] we get  $\mathcal{N}(\mathfrak{A}_K(X)) = \mathcal{Z}_K^0(X)$ . Now use (5.5) and (8.5).  $\square$

**8.9. Corollary.** *A  $k$ -dense  $G$ -variety  $X$  is  $k$ -convex if and only if  $\mathfrak{A}_k(X)$  is finite.*

From this we derive a criterion for the existence of a standard embedding:

**8.10. Corollary.** *Let  $X = G/H$  be homogeneous and assume that  $\text{Aut}^G(X) = N_G(H)/H$  does not contain a non-trivial central split torus (e.g., if  $H$  is of finite index in its normalizer). Then  $X$  is  $k$ -convex.*

*Proof.* The connected component  $\mathfrak{A}_k(X)^0$  is a split torus sitting in the center of  $\text{Aut}^G(X)$  (see [Kno96, Cor. 5.6]). Thus  $\mathfrak{A}_k(X)^0 = 1$ .  $\square$

For  $k$ -spherical varieties one can be more specific:

**8.11. Proposition.** *Let  $X = G/H$  be a homogeneous  $k$ -spherical variety. Then  $\text{Aut}^G X = N_G(H)/H$  is an elementary group whose connected split center equals  $\mathfrak{A}_k(X)^0$ . In particular  $X$  is  $k$ -convex if and only if  $N_G(H)/H$  is anisotropic.*

*Proof.* The action of an automorphism is uniquely determined by its restriction to  $X_{\text{el}}$ . Thus  $\text{Aut}^G X$  is a subquotient of  $MA$  and therefore elementary. Its largest split subtorus acts automatically by  $k$ -central automorphisms. Thus it equals  $\mathfrak{A}_k(X)^0$ . The last assertion follows from Theorem 8.8.  $\square$

Theorem 8.8 also gives a way to make any  $G$ -variety  $k$ -convex by taking the quotient by  $\mathfrak{A}_k(X)$ . More generally, the following holds:

**8.12. Lemma.** *Every  $k$ -dense  $G$ -variety  $X$  contains an open  $G \times \mathfrak{A}_k(X)$ -stable subset  $X_0$  for which the orbit space  $X_0/\mathfrak{A}_k(X)$  exists. Moreover, for any  $k$ -subgroup  $\mathfrak{A} \subseteq \mathfrak{A}_k(X)$  the quotient  $X' := X_0/\mathfrak{A}$  has the following properties:*

$$\begin{aligned}\Xi_k(X') &= \{\chi \in \Xi_k(X) \mid \text{res}_{\mathfrak{A}} \chi = 0\}, \quad A_k(X') = A_k(X)/\mathfrak{A}, \\ X'_{\text{el}} &= X_{\text{el}}/\mathfrak{A}, \quad X'_{\text{an}} = X_{\text{an}}, \quad c_k(X') = c_k(X), \quad \text{rk}_k(X') = \text{rk}_k X - \dim \mathfrak{A} \\ \mathcal{N}_k(X') &= \mathcal{N}_k(X)/\mathcal{N}(\mathfrak{A}), \quad \mathcal{Z}_k(X') = \mathcal{Z}_k(X)/\mathcal{N}(\mathfrak{A})\end{aligned}$$

*Proof.* By [Kno96, Cor. 5.4], there is an open  $G$ -stable subset of  $X$  on which the birational action of  $\mathfrak{A}_k(X)$  is regular. Then the existence of  $X_0$  is a consequence of Rosenlicht's theorem.

The first two properties of  $X'$  follow from (8.4). That formula also implies the compatibility of the  $\mathfrak{A}$ -action on  $X_{\text{el}}$  with the action induced from that of  $A_k(X)$ . This shows  $X'_{\text{el}} = X_{\text{el}}/\mathfrak{A}$  which implies all other assertions except the last one on that valuation cone. That follows from its  $K$ -counterpart [Kno93, Satz 8.1 (4.)].  $\square$

**8.13. Corollary.** *Let  $X$  be a  $k$ -dense  $G$ -variety and let  $\mathfrak{A} \subseteq \mathfrak{A}_k(X)$  be an open  $k$ -subgroup. Then there is a  $G \times \mathfrak{A}$ -stable open dense subset  $X_0 \subseteq X$  such that  $X_0/\mathfrak{A}$  has a standard embedding.*

Now we study another important class of  $G$ -varieties which generalize rank-0-varieties:

**8.14. Definition.** A  $k$ -dense  $G$ -variety  $X$  is called  *$k$ -horospherical* if  $\mathcal{Z}_k(X) = \mathcal{N}_k(X)$ .

One reason for the importance of  $k$ -horospherical varieties is the fact that a stratum  $X(\mathcal{C})$  of an embedding  $X(\mathcal{F})$  is  $k$ -horospherical if and only if  $\mathcal{C}$  contains an inner point of  $\mathcal{Z}_k(X)$ . This follows immediately from Theorem 7.3 iv).

Theorem 8.8 implies that  $k$ -horospherical varieties are also characterized by the equality  $\mathfrak{A}_k(X) = A_k(X)$  which means that such a variety carries an action of  $A_k(X)$ .

**8.15. Theorem.** *For a  $k$ -dense  $G$ -variety  $X$  the following are equivalent:*

- i)  $X$  is  $k$ -horospherical.
- ii) An open dense  $G$ -stable subset of  $X$  is induced from an elementary action.
- iii)  $X = G \cdot X^N$ .

*Proof.*  $i) \Rightarrow ii)$  Let  $X, X_0$  and  $X'$  be as in Lemma 8.12 with  $\mathfrak{A} = A_k(X)$ . Since  $\text{rk}_k X' = 0$ , Theorem 8.3 implies that there is an open subset  $X'_1 \subseteq X'$  which is induced from an anisotropic action:  $X'_1 = G \cong^{Q^-} Y'$ . Let  $Y \subseteq X_0$  be the preimage of  $Y'$ . Because  $Y \rightarrow Y'$  is an  $A_k(X)$ -bundle the action on  $Y$  is elementary with  $X_1 = G \times^{Q^-} Y$  being open in  $X_0$ .

The proof of the implications  $ii) \Rightarrow iii) \Rightarrow i)$  are the same as that of the analogous statements of Theorem 8.3.  $\square$

For homogeneous varieties this translates into:

**8.16. Corollary.** *Let  $H \subseteq G$  be a  $k$ -subgroup and  $X = G/H$ . Then the following are equivalent:*

- i)  $X$  is  $k$ -horospherical.

ii) There is a parabolic  $Q \subseteq G$  with  $R_{\text{el}}Q \subseteq H \subseteq Q$ .

iii)  $H$  contains a  $G(k)$ -conjugate of  $N$ .

iv)  $X$  is  $k$ -spherical and  $H$  is normalized by a split torus  $A_0$  with  $\dim A_0 = \text{rk}_k X$  and  $|A_0 \cap H| < \infty$ .

*Proof.* The proof of the equivalence of the first three parts analogous to that of Corollary 8.5. Now assume that they hold. Then  $G/H$  is  $k$ -spherical since  $G/N$  is. Moreover, it is well known that the split torus  $A_k(X) \subseteq \text{Aut}_k X = N_H(X)/H$  can be lifted to an isogenous torus  $A_0 \subseteq N_G(H)$ . Whence iv). Conversely, when  $X$  is  $k$ -spherical the action of  $A_0$  on  $X$  is automatically  $k$ -central and locally effective. Thus  $\mathfrak{A}_k(X) = A_k(X)$  and  $X$  is  $k$ -horospherical.  $\square$

We derive from this a characterization of  $k$ -spherical varieties:

**8.17. Proposition.** *For a homogeneous  $k$ -dense  $G$ -variety  $X$  the following are equivalent:*

i)  $X$  is  $k$ -spherical.

ii) The number of  $k$ -dense  $G$ -orbits in any normal equivariant embedding  $X \hookrightarrow \overline{X}$  is finite.

*Proof.* One implication follows from Corollary 4.14. For the converse choose a valuation  $v$  in the interior of  $\mathcal{Z}_k(X)$  and let  $X \hookrightarrow \overline{X} = X \cup D$  be the associated embedding (with  $D = X$  if  $v = 0$ ). It follows from Theorem 7.3 that  $D$  is horospherical with  $c_k(D) = c_k(X)$ . Moreover,  $D$  contains by assumption only finitely many  $k$ -dense orbits. Since  $D$  is  $k$ -dense itself one of these orbits would be open. Then Corollary 8.16 implies that  $D$  is  $k$ -spherical. Hence  $c_k(X) = c_k(D) = 0$ .  $\square$

## 9. The Weyl group

Our next goal is to study the valuation cone  $\mathcal{Z}_k(X)$ . Our strategy is to compare it with  $\mathcal{Z}_K(X)$  which is known to be the Weyl chamber for a reflection group ([Bri90, Kno93]). But first, we compare the canonical parabolic subgroups  $Q_k(X)$  and  $Q_K(X)$ .

**9.1. Proposition.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then the conjugacy class of  $Q_K(X)$  is defined over  $k$  and  $Q_k(X) = Q_K(X)P$ .*

*Proof.* The first part is clear since  $X$  is defined over  $k$ . For the second, we apply Corollary 4.6 in two steps. First, we get a Levi subgroup  $L_k \subseteq Q_k(X)$  and an  $L_k$ -stable slice  $X_{\text{el}} \subseteq X$  such that  $Q_k(X) \times^{L_k} X_{\text{el}} \rightarrow X$  is an open embedding. Considering  $X_{\text{el}}$  as a  $K$ -variety we get in a second step a parabolic  $K$ -subgroup  $\overline{Q} \subseteq L_k$  with Levi subgroup  $L_K$  and an  $L_K$ -stable slice  $\overline{X}_{\text{el}} \subseteq X_{\text{el}}$  such that  $\overline{Q} \times^{L_K} \overline{X}_{\text{el}} \rightarrow X_{\text{el}}$  is an open embedding. Combined we get an open embedding

$$(9.1) \quad \tilde{Q} \times^{L_K} \overline{X}_{\text{el}} \rightarrow X \text{ with } \tilde{Q} := R_u Q_k \cdot \overline{Q}.$$

So uniqueness (Proposition 4.5) implies that  $Q_K(X) = \tilde{Q}$ . Next, let  $\overline{P}$  be the image of  $P$  in  $L_k$ . Then  $\overline{P}$  contains all anisotropic simple factors of  $L_k$  (see (3.2)). On the other hand, all isotropic factors act trivially on  $X_{\text{el}}$  and are thus contained in  $\overline{Q}$ . Combined, this implies  $L_k = \overline{Q} \cdot \overline{P}$  and therefore

$$(9.2) \quad Q_k(X) = R_u Q_k(X) \cdot \overline{Q} \cdot \overline{P} = Q_K(X)P. \quad \square$$

Observe that  $Q_k(X)$  equals the product  $Q_K(X)P$  as opposed to just the subgroup generated by it. We translate Proposition 9.1 into a combinatorial statement. Recall the set  $S^0 \subseteq S$  of compact roots. These are, by definition, the simple roots of  $M$ . On the other hand, the parabolic  $Q_k(X)$  is defined over  $k$  and therefore corresponds to a subset  $S_k^{(p)}(X) \subseteq S_k$  of  $k$ -parabolic roots. Replacing  $k$  by  $K$ , we also get a set  $S^{(p)}(X) \subseteq S$  of  $K$ -parabolic simple roots corresponding to  $Q_K(X)$ . These sets are related as follows:

**9.2. Corollary.** *i)  $S^{(p)}(X)$  is  $\mathcal{G}^*$ -stable (see (2.6) for the notation).*

*ii)  $S_k^{(p)}(X) = \text{res}'_A S^{(p)}(X)$  (see (2.2) for the notation).*

*iii) Let  $C$  be a connected component of  $S^{(p)}(X) \cup S^0$  in the Dynkin diagram of  $S$ . Then  $C \subseteq S^{(p)}(X)$  or  $C \subseteq S^0$ .*

*Proof.* *i)* just means that the conjugacy class  $Q_K(X)$  is defined over  $k$ . *ii)* is then equivalent to  $Q_k(X)$  being generated by  $Q_K(X)$  and  $P$ . Finally, *iii)* holds because  $Q_k(X)$  is even the product of  $Q_K(X)$  and  $P$ .  $\square$

As mentioned above, it is known that  $\mathcal{Z}_K(X)$  is the fundamental domain for a finite reflection group  $W_K(X)$  (see [Kno94, Thm. 7.4]). Following the methods of [Kno94], we show an analogous fact for  $\mathcal{Z}_k(X)$ . For this, we need the following geometric result:

**9.3. Lemma.** *Let  $X$  be a normal  $k$ -dense  $G$ -variety. Then there is a Borel subgroup  $B \subseteq P$ , a  $k$ -subgroup  $L_K \subseteq G$ , and a point  $x \in X(k)$  such that:*

*i)  $L_K$  is a Levi subgroup of  $Q_K(X)$  where  $Q_K(X)$  is the canonical parabolic subgroup of  $X$  over  $K$  which contains  $B$ .*

*ii) Let  $T \subseteq L_K$  be a maximal  $k$ -torus. Then there is a commutative diagram as follows*

$$(9.3) \quad \begin{array}{ccccc} A & \subseteq & T & \subseteq & L_K \\ \downarrow & & \downarrow & & \downarrow \searrow \varphi \\ A_k(X) & \subseteq & A_K(X) & \xrightarrow{\sim} & Tx \subseteq X \end{array}$$

where  $\varphi$  is the orbit map  $g \mapsto gx$ . With the induced  $k$ -structure of  $A_K(X)$  all maps are defined over  $k$ .

*iii) The action of the little Weyl group  $W_K(X)$  on  $A_K(X)$  extends to the closure  $\overline{Tx}$  in  $X$ .*

*iv) Let  $a : \mathbf{G}_m \rightarrow A_K(X)$  be a 1-parameter  $K$ -subgroup which we consider as an element of  $\mathcal{N}_K(X)$ . Let  $w \in W_K(X)$  with  $wa \in \mathcal{Z}_K(X)$  and let  $v$  be the  $K$ -central valuation corresponding to  $wa$ . Then  $x_0 := \lim_{t \rightarrow 0} a(t)x$  exists in  $X$  if and only if  $v$  has a center  $Y$  in  $X$  and in that case  $x_0 \in Y$ .*

*Proof.* First, by an affine cone construction as in the proof of Theorem 4.2 it is easy to see that we may assume  $X$  to be quasiaffine. Then  $X$  will be non-degenerate in the sense of [Kno94, §3].

We recall some facts from [Kno94]. Choose a Borel subgroup  $B \subseteq P$  which determines a canonical  $K$ -parabolic  $Q_K = Q_K(X) \supseteq B$ . Let  $U_K := Q_{K,u}$  be its unipotent radical. Let

$\pi : T_X^* \rightarrow X$  be the cotangent bundle and  $m : T_X^* \rightarrow \mathfrak{g}^*$  its moment map. For  $\alpha \in T_X^*$  let  $x_\alpha := \pi(\alpha) \in X$  and  $L_\alpha := C_G(m(\alpha))$ . Then in [Kno94] it was shown that

$$(9.4) \quad m^{-1}(\mathfrak{u}_K^\perp) = \{\alpha \in T_X^* \mid \alpha(\mathfrak{u}_K x_\alpha) = 0\}$$

contains an open  $Q_K$ -stable subset  $C$  such that  $\pi : C \rightarrow X$  is dominant and for all  $\alpha \in C$  the pair  $L_K = L_\alpha, x = x_\alpha$  has all the properties *i)–iv)* of the theorem except for:

- $L_K$  and  $T$  might not be defined over  $k$  and therefore
- the left square of diagram (9.3) may not exist.

More precisely, *i)* holds by [Kno94, Thm. 2.3 and §3]. For *ii)* see [Kno94, §4]. Assertion *iii)* follows from [Kno94, Cor. 6.3] and *iv)* from [Kno94, Thm. 7.3].

The two remaining properties above will also hold if  $\alpha$  would be a  $k$ -rational point of  $T_X^*$ . We claim that such  $\alpha$  is indeed possible to find after possibly changing  $B$  by a conjugate in  $P$ .

Do see this, we first we apply Corollary 4.6 to  $X$ . This way, we get a decomposition  $Q_k := Q_k(X) = L_k U_k$  and an  $L_k$ -subvariety  $X_{\text{el}} \subseteq X$  such that  $U_k \times X_{\text{el}} \rightarrow X$  is an open embedding. Next we apply Corollary 4.6 to the  $L_k$ -variety  $X_{\text{el}}$  but now considered as a variety over  $K$ . We get a decomposition  $\tilde{Q}_K := Q_K(X_{\text{el}}) = \tilde{L}_K \tilde{U}_K \subseteq L_k$  and an  $\tilde{L}_K$ -subvariety  $\tilde{X} \subseteq X_{\text{el}}$  such that  $\tilde{U}_K \times \tilde{X} \rightarrow X_{\text{el}}$  is an open embedding. Since then also  $U_k \times \tilde{U}_K \times \tilde{X} \rightarrow X$  is an open embedding, uniqueness (Proposition 4.5) implies that  $Q_K = \tilde{Q}_K U_k$  and  $U_K = U_k \tilde{U}_K$ .

Now consider the restriction map

$$(9.5) \quad \pi^{-1}(X_{\text{el}}) = T_X^*|_{X_{\text{el}}} \rightarrow T_{X_{\text{el}}}^*.$$

It has an  $L_k$ -equivariant section by extending  $\alpha \in T_{X_{\text{el}}}^*$  by 0 on  $\mathfrak{u}_k x_\alpha$ . In other words, we get a  $k$ -isomorphism

$$(9.6) \quad \iota : \{\alpha \in T_X^* \mid x_\alpha \in X_{\text{el}}, \alpha(\mathfrak{u}_k x_\alpha) = 0\} \rightarrow T_{X_{\text{el}}}^*$$

Now let

$$(9.7) \quad C_{\text{el}} := C \cap \pi^{-1}(X_{\text{el}}) \subseteq \{\alpha \in T_X^* \mid x_\alpha \in X_{\text{el}}, \alpha(\mathfrak{u}_K x_\alpha) = 0\}.$$

Then  $C_{\text{el}} \rightarrow X_{\text{el}}$  is dominant since  $C \rightarrow X$  is dominant and  $U_k \cdot X_{\text{el}}$  is dense in  $X$ . Because of  $\mathfrak{u}_K = \mathfrak{u}_k \oplus \tilde{\mathfrak{u}}_K$ , the morphism  $\iota$  maps the set  $C$  maps isomorphically to an open subset of

$$(9.8) \quad \{\alpha \in T_{X_{\text{el}}}^* \mid \alpha(\tilde{\mathfrak{u}}_K x_\alpha) = 0\}$$

or, more precisely, to the irreducible component  $\tilde{C}$  which maps dominantly to  $X_{\text{el}}$ . Now we can apply [Kno94, Theorem 3.2] to the  $L_k$ -variety  $X_{\text{el}}$  and conclude that the morphism

$$(9.9) \quad L_k \times \tilde{C} \rightarrow T_{X_{\text{el}}}^* :$$

is dominant. Because of  $L_k = (P \cap L_k) \cdot (L_k)_{\text{el}}$  (see (3.2)) and because  $(L_k)_{\text{el}}$  acts trivially on  $X_{\text{el}}$  we see that also

$$(9.10) \quad (P \cap L_k) \times C_{\text{el}} \rightarrow T_{X_{\text{el}}}^* : (g, \alpha) \mapsto g\iota(\alpha)$$

is dominant. Now recall that  $X$ , and therefore  $X_{\text{el}}$  and  $T_{X_{\text{el}}}^*$  are  $k$ -dense. Thus, we can find  $g \in P \cap L_k$  and  $\alpha \in C_{\text{el}}$  such that  $g\alpha \in T_{X_{\text{el}}}^*(k)$ . By replacing  $B$  with  $gBg^{-1}$  we may

assume that  $g = 1$ . Then  $\alpha$  is a  $k$ -rational point in  $C_{\text{el}}$ , hence in  $C$ . Observe that because of  $g \in P \cap L_k$  none of  $P$ ,  $Q_k$ , and  $L_k$  change.

Now both  $L_K := L_\alpha$  and  $x := x_\alpha$  are defined over  $k$ . Then we can choose a maximal torus  $T \subseteq P \cap L_K$  which automatically contains the center of  $L_k$  and therefore  $A$ . From this and the fact that  $x \in X_{\text{el}}$  we get the left square in [ii](#)).  $\square$

**9.4. Remark.** An  $A$ -orbit  $Ax$  as above is called a  $k$ -flat. Accordingly the  $T$ -orbit  $Tx$  is a  $K$ -flat. In general, there are multiparameter families of flats which are not conjugate to each other. Observe that the set of flats which have the last two properties [iii](#)) and [iv](#)) depends on the global structure of  $X$  (and not just its generic structure). For example, one could make any specific flat “bad” by removing the orbit  $Gx_0$  from  $X$  (unless  $Y$  has an open orbit) or blowing it up. This problem does not occur for spherical varieties.

Now we are in the position to show that also  $\mathcal{Z}_k(X)$  is the Weyl chamber of a reflection group. For this we define

$$(9.11) \quad N(\mathcal{N}_k) := \{w \in W_K(X) \mid w\mathcal{N}_k(X) = \mathcal{N}_k(X)\}$$

to be the normalizer of  $\mathcal{N}_k(X)$  in  $W_K(X)$ ,

$$(9.12) \quad C(\mathcal{N}_k) := \{w \in W_K(X) \mid w|_{\mathcal{N}_k(X)} = \text{id}_{\mathcal{N}_k(X)}\},$$

its centralizer, and

$$(9.13) \quad W_k(X) := N(\mathcal{N}_k)/C(\mathcal{N}_k),$$

the restricted Weyl group.

**9.5. Theorem.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then  $\mathcal{Z}_k(X)$  is a fundamental domain for the action of  $W_k(X)$  on  $\mathcal{N}_k(X)$ .*

*Proof.* First of all, because of  $\mathcal{Z}_k \subseteq \mathcal{Z}_K$  no two different elements of  $\mathcal{Z}_k(X)$  are  $W_k(X)$ -conjugate because the same is true for  $\mathcal{Z}_K(X)$  and  $W_K(X)$ . It remains to show, that every element of  $\mathcal{N}_k(X)$  is  $W_k(X)$ -conjugate to an element of  $\mathcal{Z}_k(X)$ .

We claim that it suffices to show that every  $v \in \mathcal{N}_k(X)$  is  $W_K(X)$ -conjugate to an element of  $\mathcal{Z}_k(X)$ . To see this, define

$$(9.14) \quad \mathcal{N}_k^\circ := \mathcal{N}_k(X) \setminus \bigcup_{w \in W_K(X) \setminus N(\mathcal{N}_k)} w\mathcal{N}_k(X).$$

This is an open Zariski dense subset of  $\mathcal{N}_k(X)$  with the property that  $w \in W_K(X)$ ,  $v \in \mathcal{N}_k^\circ$ , and  $wv \in \mathcal{N}_k(X)$  imply  $w \in N(\mathcal{N}_k)$ . Thus, if for every  $v \in \mathcal{N}_k^\circ$  there is  $w \in W_K(X)$  with  $wv \in \mathcal{N}_k(X)$  then automatically  $w \in N(\mathcal{N}_k)$ . The claim follows by continuity.

So let  $v \in \mathcal{N}_k^\circ$ , let  $v_0 \in \mathcal{Z}_K$  be the unique element in the  $W_K(X)$ -orbit of  $v$ , and let  $(\overline{X}, D)$  be a model of  $v_0$  (not necessarily defined over  $k$ ). Let further  $a : \mathbf{G}_m \rightarrow A_k$  be a cocharacter in the ray spanned by  $v$ . Then according to [Lemma 9.3](#), the limit  $x_0 = \lim_{t \rightarrow 0} a(t)x$  exists and lies in  $D$ . In particular,  $D$  contains the  $k$ -rational point  $x_0$ . We claim that that this implies that  $D$  is in fact  $k$ -dense. For this let  $Z$  be its Zariski closure of  $D(k)$  in  $D$  and put  $\overline{X}' := \overline{X} \setminus Z$  and  $D' := D \setminus Z$ . If  $Z \neq D$  then  $(\overline{X}', D')$  would be still a model for  $v_0$  but this time with  $D'(k) = \emptyset$ . But this is impossible by the argument above applied to  $\overline{X}'$  instead of  $\overline{X}$ .

Thus  $v_0$  is  $k$ -dense and  $K$ -central, hence  $v_0 \in \mathcal{Z}_k$  by [Theorem 6.6](#).  $\square$

Recall that a facet of a convex cone is a face of codimension one.

**9.6. Corollary.** *The group  $W_k(X)$  is generated by reflections about the facets of  $\mathcal{Z}_k(X)$ .*

*Proof.* Indeed, only reflection groups have closed fundamental domains.  $\square$

Fundamental domains of a finite reflection group are very special:

**9.7. Corollary.** *The valuation cone  $\mathcal{Z}_k(X)$  is cosimplicial, i.e., it is defined by a set of linearly independent linear inequalities.*

**9.8. Corollary.** *Let  $X(\mathcal{F})$  be the embedding corresponding to a fan  $\mathcal{F}$ . Then there is an  $M$ -stable open dense subset  $X_{\text{an}}^0 \subseteq X_{\text{an}}$  such that for all  $y \in X_{\text{an}}^0(k)$  the following holds: let  $F(y) \subseteq X_{\text{el}}$  be the  $A_k(X)$ -orbit over  $y$  and  $\mathfrak{q}(y)$  its closure in  $X$ . Then  $\mathfrak{q}(y)$  is isomorphic to the toroidal  $A_k(X)$ -embedding which corresponds to the fan*

$$(9.15) \quad W_k \mathcal{F} := \{w\mathcal{C} \mid w \in W_k(X), \mathcal{C} \in \mathcal{F}\}.$$

*In particular,  $\overline{A}$  carries an action of  $W_k(X)$  and is complete if and only if  $\mathcal{F}$  is complete.*

*Proof.* The same proof as that for [Kno94, Cor. 6.2] shows that there is an open subset  $X_{\text{an}}^0$  such that for all  $y \in X_{\text{an}}^0(k)$  the embeddings  $\mathfrak{q}(y)$  are isomorphic to each other. This common  $A_k(X)$ -embedding corresponds to a fan  $\mathcal{F}'$  supported in  $\mathcal{N}_k(X)$ . Lemma 9.3iv) implies that  $\mathcal{F}'$  is  $W_k(X)$ -stable. The construction of  $X(\mathcal{F})$  shows that  $A_k(\mathcal{F})$  is an open subset of  $A_k(\mathcal{F}')$  which means that  $\mathcal{F}$  is a subfan of  $\mathcal{F}'$ , wence  $W_k \mathcal{F} \subseteq \mathcal{F}$ . Equality follows from the fact that  $\mathcal{Z}_k(X)$  is a fundamental domain for  $W_k(X)$ .  $\square$

Another consequence of Lemma 9.3 is:

**9.9. Proposition.** *Let  $X$  be a  $k$ -convex,  $k$ -spherical, and quasiprojective  $G$ -variety. Then  $X$  is affine and homogeneous.*

*Proof.* Let  $X_0 \subseteq X$  be the open  $G$ -orbit and let  $X \hookrightarrow \overline{X}$  be an equivariant affine embedding. Suppose  $X_0 \neq \overline{X}$ . Then Kempf's Theorem 3.6 implies that  $\overline{X} \setminus X_0$  contains a  $k$ -rational point  $y$ . Lemma 6.9 shows that  $Y := \overline{Gy}$  is the center of a  $k$ -central valuation  $v \in \mathcal{Z}_k(X)$ . Now let  $A_k x \subseteq A_K x \subseteq X_0$  be a flat as in Lemma 9.3 and let  $\overline{A}$  be the closure of  $A_k x$  in  $\overline{X}$ . Then  $\overline{A}$  meets  $Y$  by part iv) of Lemma 9.3. Therefore, the normalization of  $\overline{A}$  is a non-trivial affine torus embedding and therefore which given by a non-zero strictly convex cone  $\mathcal{C} \subseteq \mathcal{N}_k$ . Moreover,  $\mathcal{C}$  is  $W_k$ -stable by part iii). Such a cone can only exist when the fixed point set  $\mathcal{N}_k^{W_k}$  is non-zero. But that set equals  $\mathcal{Z}_k^0$  which is 0, by assumption. This contradiction shows that  $X = X_0 = \overline{X}$  is homogeneous and affine.  $\square$

Here is an application:

**9.10. Corollary.** *Let  $X = G/H$  be a  $k$ -spherical  $G$ -variety. Assume that  $N_G(H)/H$  is anisotropic (e.g. finite) and that  $H$  is unimodular. Then  $H$  is reductive.*

*Proof.* The condition on  $N_G(H)/H$  means that  $X$  is  $k$ -convex (Proposition 8.11). Then also  $G/H^0$  is  $k$ -convex (Corollary 6.10) and  $k$ -spherical. Thus, we may assume without loss of generality that  $H$  is connected.

Now consider  $\mathfrak{h} = \text{Lie } H$  and let  $\tau$  be a generator of the line  $\wedge^d \mathfrak{h}$  (with  $d = \dim \mathfrak{h}$ ). Let  $N_\tau$  be its isotropy group in  $N_G(H)$ . Then  $H$  being unimodular means that  $H \subseteq N_\tau$ . Since, by assumption,  $N_G(H)/H$  is anisotropic also its subgroup  $N_\tau/H$  is anisotropic.



Therefore it acts (locally) trivially on the fibers of  $X_{\text{el}} \rightarrow X_{\text{an}}$  which entails  $\text{rk}_k G/H = \text{rk}_k X_{\text{el}} = \text{rk}_k X_{\text{el}}/N_\tau = \text{rk}_k G/N_\tau$ . From Corollary 6.10 we get  $\mathcal{Z}_k(G/H) = \mathcal{Z}_k(G/N_\tau)$  and, in particular, that  $G/N_\tau$  is  $k$ -convex, as well.

We now consider  $\tau$  as a point of  $\wedge^d \mathfrak{g}$ . The orbit map then yields an embedding

$$(9.16) \quad \varrho : G/N_\tau \hookrightarrow \wedge^d \mathfrak{g} : gN_\tau \mapsto g\tau$$

which means that  $G/N_\tau$  is quasiaffine. Since it is also  $k$ -spherical and  $k$ -convex we get from Proposition 9.9 that  $G/N_\tau$  is affine which implies that  $N_\tau$  is reductive. But then its normal subgroup  $H$  is reductive, as well.  $\square$

## 10. The root system

In this section,  $X$  will be a fixed  $k$ -dense  $G$ -variety. Accordingly, if there is no danger of confusion we drop the reference to  $X$  in most notation:

$$(10.1) \quad S^{(p)}, S_k^{(p)}, \Xi_k, A_k, \mathcal{N}_k, \mathcal{Z}_k, W_k$$

where  $k$  can also be replaced by  $K$ .

After assigning a  $k$ -Weyl group to  $X$  we now construct an integral root system for it. The simple roots should vanish on the facets of the Weyl chamber  $\mathcal{Z}_k$  and are therefore unique up to a positive scalar. So, we face an issue of normalization. The most obvious is:

**10.1. Definition.** Let  $X$  be  $k$ -dense  $G$ -variety.

*i)* A *primitive  $k$ -spherical root of  $X$*  is a primitive element  $\sigma \in \Xi_k$  such that  $\mathcal{Z}_k \cap \{\sigma \geq 0\}$  is a facet of  $\mathcal{Z}_k$ . In other words,  $\sigma$  is the unique shortest element of  $\Xi_k$  which is an outward normal vector to a facet of  $\mathcal{Z}_k$ .

*ii)* The set of primitive  $k$ -spherical roots is denoted by  $\Sigma_k^{\text{pr}} := \Sigma_k^{\text{pr}}(X)$ . It is in 1:1-correspondence with the facets of  $\mathcal{Z}_k$ .

*iii)* The *primitive  $k$ -root system of  $X$*  is

$$(10.2) \quad \Phi_k^{\text{pr}} = \Phi_k^{\text{pr}}(X) := W_k \cdot \Sigma_k^{\text{pr}} \subseteq \Xi_k.$$

Corollary 9.7 implies that  $\Sigma_k^{\text{pr}}$  is linearly independent and that the valuation cone has the presentation

$$(10.3) \quad \mathcal{Z}_k = \{a \in \mathcal{N}_k \mid \sigma(a) \leq 0 \text{ for all } \sigma \in \Sigma_k^{\text{pr}}\}.$$

Recall the set  $S_k \in \Xi(A)$  of restricted simple roots of  $G$ . Then Proposition 5.4 has the following reformulation:

**10.2. Lemma.** Every  $\sigma \in \Sigma_k^{\text{pr}}$  is a linear combination  $\sigma = \sum_{\alpha \in S_k} c_\alpha \alpha$  with  $c_\alpha \in \mathbb{Q}_{\geq 0}$ .

The set of  $\alpha \in S_k$  with  $c_\alpha > 0$  is called the *support of  $\sigma$*  (denoted  $\text{supp } \sigma$ ).

A *weight lattice* for a root system  $\Phi$  with Weyl group  $W$  is a lattice  $\Xi$  containing  $\Phi$  with  $(1 - s_\sigma)\Xi \subseteq \mathbb{Z}\sigma$  for all  $\sigma \in \Phi$ . Here,  $s_\sigma$  denotes the reflection about  $\sigma$ . If  $\Phi$  is a reduced root system then this is equivalent to  $W$  acting trivially on  $\Xi/\langle \Phi \rangle_{\mathbb{Z}}$ . This follows easily from  $\mathbb{Q}\sigma \cap \langle \Phi \rangle_{\mathbb{Z}} = \mathbb{Z}\sigma$ . Yet another criterion is the existence of a (unique) coroot  $\sigma^\vee \in \text{Hom}(\Xi, \mathbb{Z})$  with

$$(10.4) \quad s_\sigma(\chi) = \chi - \langle \chi | \sigma^\vee \rangle \sigma \text{ for all } \chi \in \Xi.$$

We record:

**10.3. Proposition.**  $\Phi_k^{\text{pr}}$  is a reduced root system with Weyl group  $W_k$ . The set  $\Sigma_k^{\text{pr}}$  is a system of simple roots for  $\Phi_k^{\text{pr}}$ . The valuation cone  $\mathcal{Z}_k$  is the antidominant Weyl chamber with respect to  $\Sigma_k^{\text{pr}}$ . The lattice  $\Xi_k$  is a weight lattice for  $\Phi_k^{\text{pr}}$ .

*Proof.* Everything follows from the definitions and the fact that  $\mathcal{Z}_k$  is a fundamental domain for  $W_k$  once we know that  $\Xi_k$  is  $W_k$ -stable. Over  $K$  this is [Kno94, Thm. 4.2]. For  $k$  it now follows from (4.24).  $\square$

Of course, all constructions above are valid with  $k$  replaced by  $K$ . In this case, several other normalizations of spherical root are in use (see, for example, the account in [VS13]) but the primitive one is most common. Therefore, over  $K$  we are going to drop the superscript  $\text{pr}$  and write  $\Sigma_K = \Sigma_K(X) := \Sigma_K^{\text{pr}}$  and  $\Phi_K = \Phi_K(X) := \Phi_K^{\text{pr}}$ . In the following, we will focus on the relationship between  $\Sigma_K$  and  $\Sigma_k^{\text{pr}}$ .

For any  $\sigma \in \Xi_K$  let  $\bar{\sigma} := \text{res}_A \sigma$  be the restriction to  $A$  and put

$$(10.5) \quad \Xi_K^0 = \Xi_K^0(X) = \{\sigma \in \Xi_K(X) \mid \bar{\sigma} = 0\}.$$

Elements of

$$(10.6) \quad \Sigma_K^0 = \Sigma_K^0(X) := \Sigma_K \cap \Xi_K^0 = \{\sigma \in \Sigma_K \mid \bar{\sigma} = 0\}.$$

will be called *(k-)compact spherical roots*. They comprise the set From Lemma 10.2 (over  $K$ ) it follows that the compact spherical roots can be recovered from the compact simple roots:

$$(10.7) \quad \Sigma_K^0 = \{\sigma \in \Sigma_K \mid \text{supp}(\sigma) \subseteq S^0\}.$$

We will see later (Corollary 10.16) that  $\Sigma_K^0$  is the set of  $K$ -spherical roots of  $X_{\text{el}}$ . Let

$$(10.8) \quad W_K^0 = W_K^0(X) = \langle s_\sigma \mid \sigma \in \Sigma_K^0 \rangle \subseteq W_K$$

be the subgroup of  $W_K$  which is generated by all reflections about compact spherical roots. Using notation (2.2), we define the set of restricted spherical roots as

$$(10.9) \quad \Sigma_k = \Sigma_k(X) := \text{res}'_A \Sigma_K = \{\bar{\sigma} \mid \sigma \in \Sigma_K, \bar{\sigma} \neq 0\}.$$

Our goal is to show that  $\Sigma_k$  is another system of simple roots for  $W_k$ .

**10.4. Remark.** In general,  $\Sigma_k$  and  $\Sigma_k^{\text{pr}}$  are different. This already happens in the group case  $X = H \times H/H$  with  $H = U(1, 1)$  and  $k = \mathbb{R}$ . In that case, the restricted simple root  $\bar{\sigma}$  is not primitive. Thus  $\Sigma_k = \{\pm \bar{\sigma}\}$  while  $\Sigma_k^{\text{pr}} = \{\pm \frac{1}{2} \bar{\sigma}\}$ .

Next we study the action of the Galois group  $\mathcal{G} = \text{Gal}(K|k)$ . Recall the situation of Lemma 9.3, in particular the maximal  $k$ -torus  $T$  containing the maximal  $k$ -split torus  $A$  and a point  $x \in X(k)$  such that the orbit  $Tx$  is a flat. Then the following objects will be considered:

$$(10.10) \quad \begin{array}{ccc} \Sigma_K & \subseteq & \Xi_K \longrightarrow \Xi_k \\ & \cap & \cap \\ S & \subseteq & \Xi(T) \longrightarrow \Xi(A) \end{array}$$

The Galois group  $\mathcal{G}$  acts on  $\Xi(T)$  and trivially on  $\Xi(A)$ . Since  $Tx \cong A_K$  is defined over  $k$ , the sublattices  $\Xi_K$  and  $\Xi_k$  are  $\mathcal{G}$ -stable.

Now recall the  $*$ -action of  $\mathcal{G}$  on  $\Xi(T)$ . It is defined as

$$(10.11) \quad \gamma * \chi = w_\gamma \gamma \chi$$

where  $w_\gamma$  is the unique element of  $W(G)$  with  $w_\gamma \gamma S = S$ . It is known that  $w_\gamma \in W^0 = \langle s_\alpha \mid \alpha \in S^0 \rangle$  (see §2). This implies that the restriction  $\text{res}_A$  is also  $\mathcal{G}^*$ -invariant.

The subgroup  $\Xi_K \subseteq \Xi(T)$  is  $\mathcal{G}^*$ -stable, as well. In fact, if  $f$  is a  $B$ -semiinvariant function then  ${}^{n_\gamma} f$  is also  $B$ -semiinvariant with character  $\gamma * \chi_f$ . Here,  $n_\gamma \in N_G(T)$  is a lift of  $w_\gamma$ .

We define a  $\mathcal{G}$ -action on valuations by  $\gamma v(f) := v(\gamma^{-1} f)$ . Then it is easy to see that the injection  $\iota_K : \mathcal{Z}_K \hookrightarrow \mathcal{N}_K$  is  $\mathcal{G}^*$ -equivariant. Indeed, because of the  $G$ -invariance of  $v$  we have

$$(10.12) \quad \iota_K(\gamma v)(\chi) = \gamma v(f_\chi) = v(\gamma^{-1} f_\chi) = v({}^{n_\gamma} f_\chi) = v(f_{\gamma^{-1} * \chi}) = (\gamma * \iota_K(v))(\chi)$$

This shows that  $\Sigma_K$  is a  $\mathcal{G}^*$ -stable subset of  $\Xi_K$ . In particular,  $\Phi_K$  and  $W_K$  are  $\mathcal{G}^*$ -stable.

**10.5. Lemma.** *Let  $\gamma \in \mathcal{G}$ . Then  $w_\gamma \Xi_K = \Xi_K$  and restriction of  $w_\gamma$  to  $\Xi_K$  is an element of  $W_K^0$ .*

*Proof.* The first assertion follows from the fact that  $\Xi_K$  is both  $\gamma$ - and  $\gamma^*$ -stable.

Next we show that there is an element  $\bar{w}_\gamma \in W_K$  with  $w_\gamma v = \bar{w}_\gamma v$  for all  $v \in \mathcal{N}_K$ . For this, choose  $v \in \mathcal{Z}_K$  with model  $(\bar{X}, D)$ . Without loss of generality we may assume that  $\bar{X}$  does not contain any other  $G$ -stable divisor  $D' \neq D$  with  $v_{D'} \in \mathcal{Z}_K$ . Consider the closure  $\overline{Tx}$  of the flat  $Tx$  in  $\bar{X}$ . Then Lemma 9.3 implies that  $\overline{Tx}$  equals the toroidal embedding  $A_K(\mathcal{F})$  where  $\mathcal{F}$  is the fan consisting of  $\{0\}$  and all rays  $\mathbb{Q}_{\geq 0} wv$  with  $w \in W_K$ . Now we twist  $\bar{X}$  by an element  $\gamma \in \mathcal{G}$ . Then the embedding  $X \hookrightarrow \gamma \bar{X} = X \cup \gamma D$  corresponds to the valuation  $\gamma * v \in \mathcal{Z}_K$ . Using the fact that  $Tx$  is  $\mathcal{G}$ -stable and applying Lemma 9.3 to  $\gamma X$  we get

$$(10.13) \quad \gamma \bigcup_w \mathbb{Q}_{\geq 0} wv = \bigcup_w \mathbb{Q}_{\geq 0} w(\gamma * v)$$

Since  $\mathcal{G}$  preserves the primitive generators of the rays we see that the orbit of  $v$  is mapped to the orbit of  $\gamma * v$ :

$$(10.14) \quad \gamma(W_K v) = W_K(\gamma * v)$$

Every  $v \in \mathcal{N}_K$  is  $W_K$ -conjugate to an element of  $\mathcal{Z}_K$ , hence (10.14) holds for all  $v \in \mathcal{N}_K$ . Now using the fact that  $W_K$  is normalized by  $\mathcal{G}^*$  we obtain

$$(10.15) \quad w_\gamma W_K v = (w_\gamma \gamma)(\gamma^{-1} W_K v) = \gamma * W_K(\gamma^{-1} * v) = W_K v$$

Thus for every  $v$  there is  $\bar{w}_\gamma \in W_K$  with  $w_\gamma v = \bar{w}_\gamma v$ . The element  $\bar{w}_\gamma$  might depend on  $v$ . It is unique for  $v$  in the interior  $\mathcal{Z}^\circ$  of  $\mathcal{Z}_K$  and there it depends continuously on  $v$ . Thus  $\bar{w}_\gamma$  is constant for  $v \in \mathcal{Z}^\circ$ , hence constant everywhere by linearity.

Because of  $\gamma \sigma = \bar{w}_\gamma^{-1} \gamma * \sigma$  we see that  $\Phi_K$  is  $\mathcal{G}$ -stable and that  $\bar{w}_\gamma$  is the unique element of  $W_K$  with

$$(10.16) \quad \bar{w}_\gamma \gamma \Sigma_K = \Sigma_K$$

We claim that  $\bar{w}_\gamma \in W_K^0$ . Let  $\Phi^0 = W_K^0 \Sigma_K^0$  be the subroot system generated by  $\Sigma_K^0$ . It follows from Lemma 10.2 that  $\Phi^0 = \{\sigma \in \Phi_K \mid \text{res}_A \sigma = 0\}$ . Since  $\text{res}_A$  is  $\mathcal{G}$ -invariant this implies that  $\Phi_K^0$  is  $\mathcal{G}$ -stable. Thus, for every  $\gamma \in \mathcal{G}$  there is a unique element  $w_\gamma^0 \in W_K^0$  with

$$(10.17) \quad w_\gamma^0 \gamma \Sigma_K^0 = \Sigma_K^0.$$

For any non-compact  $\sigma \in \Sigma_K$  let  $\tilde{\sigma} := w_\gamma^0 \gamma \sigma$ . Since  $\text{res}_A$  is  $\mathcal{G}$ - and  $W_K^0$ -invariant we get  $\text{res}_A \tilde{\sigma} = \text{res}_A \sigma$ . This and  $\mathcal{Z}_k \subseteq \mathcal{Z}_K$  imply that the spherical root  $\tilde{\sigma}$  has at least one negative value on  $\mathcal{Z}_K$  and is therefore a positive root. Combined, this shows that  $w_\gamma^0 \gamma \Sigma_K$  consists entirely of positive roots and is therefore equal to  $\Sigma_K$ . Uniqueness implies  $\bar{w}_\gamma = w_\gamma^0 \in W_K^0$ .  $\square$

The following was proved in the course of the proof:

**10.6. Corollary.** *The root system  $\Phi_K$  is a  $\mathcal{G}$ -stable subset of  $\Xi_K$  and  $W_K$  is normalized by  $\mathcal{G}$ .*

We also get that the ordinary and the  $*$ -action of  $\mathcal{G}$  are the same modulo compact roots:

**10.7. Corollary.** *Let  $\gamma \in \mathcal{G}$  and  $\chi \in \Xi_K$ . Then  $\gamma \chi - \gamma * \chi \in \langle \Sigma_K^0 \rangle_{\mathbb{Z}}$ .*

*Proof.* Because of  $\bar{w}_\gamma \in W_K^0$  we have

$$(10.18) \quad \gamma \chi - \gamma * \chi = (1 - \bar{w}_\gamma)(\gamma \chi) \in \langle \Sigma_K^0 \rangle_{\mathbb{Z}} \quad \square$$

As a consequence we get that  $\mathcal{N}_k$  can be computed from  $\Sigma_K^0$  and the  $\mathcal{G}^*$ -action on  $\Xi_K$ :

**10.8. Corollary.** *The subspace  $\mathcal{N}_k$  of  $\mathcal{N}_K$  is defined by the equations*

$$(10.19) \quad \{\sigma = 0 \mid \sigma \in \Sigma_K^0\} \cup \{\chi - \gamma * \chi = 0 \mid \chi \in \Xi_K, \gamma \in \mathcal{G}\}.$$

*Proof.* Because of  $\mathcal{N}_k = \mathcal{N}_K^{\mathcal{G}}$ , the space  $\mathcal{N}_k$  is defined by the equations  $\chi - \gamma \chi = 0$  with  $\chi \in \Xi_K$ ,  $\gamma \in \mathcal{G}$ . Moreover, all equations  $\sigma = 0$  for  $\sigma \in \Sigma_K^0$  hold on  $\mathcal{N}_k$  by definition. Modulo those equations, we have

$$(10.20) \quad \chi - \gamma \chi = \chi - \gamma * \chi$$

proving the assertion.  $\square$

Observe that the weight lattice  $\Xi_k$  can also be computed since it is the image of  $\Xi_K$  in  $\mathcal{N}_k^*$ . The equations for  $\mathcal{N}_k$  can be made a bit more precise. We start with unique  $W_k$ -stable decomposition  $\mathcal{N}_k = \mathcal{N}_k^0 \oplus \mathcal{N}_k^1$  with  $\mathcal{N}_k^0 := \mathcal{N}_k^{W_k} = \mathcal{Z}_k^0$  and such that  $\Sigma_k$  forms a basis of  $(\mathcal{N}_k^1)^*$ . Then

$$(10.21) \quad \mathcal{N}_k^0 = (\mathcal{N}_K^0)^{\mathcal{G}} = (\mathcal{N}_K^0)^{\mathcal{G}^*}.$$

Moreover, it suffices to let  $\chi$  in (10.19) run through a set of generators. Thus, the subspace  $\mathcal{N}_k^1 \subseteq \mathcal{N}_K^1$  is defined by the equations  $\sigma = 0$  for all compact spherical roots and  $\sigma - \tau = 0$  for all pairs of non-compact spherical roots which are in the same  $\mathcal{G}^*$ -orbit. This shows:

**10.9. Corollary.** *The set  $\Sigma_k \subseteq \Xi_k$  is linearly independent. Moreover,  $\bar{\sigma} = \bar{\tau} \neq 0$  for  $\sigma, \tau \in \Sigma_K$  if and only if  $\sigma$  and  $\tau$  are in the same  $\mathcal{G}^*$ -orbit.*

From this, we obtain:

**10.10. Corollary.** *There is a map  $\Sigma_k^{\text{pr}} \rightarrow \mathbb{Z}_{>0} : \sigma \mapsto n_\sigma$  such that  $\Sigma_k = \{n_\sigma \sigma \mid \sigma \in \Sigma_k^{\text{pr}}\}$ .*

*Proof.* It follows from (5.5) that  $\mathcal{Z}_k$  is also defined by the inequalities  $\bar{\sigma} \leq 0$  with  $\sigma \in \Sigma_K$ . Since they are linearly independent, they form a minimal set of inequalities. Consequently, every  $\bar{\sigma}$  is an integral multiple of an element of  $\Sigma_k^{\text{pr}}$ .  $\square$

Since elements of  $\Sigma_K$  and  $\Sigma_k$  correspond to facets of  $\mathcal{Z}_K$  and  $\mathcal{Z}_k$ , respectively, we obtain:

**10.11. Corollary.** *Let  $\mathcal{F} \subseteq \mathcal{Z}_K$  be a facet and  $\overline{\mathcal{F}} := \mathcal{F} \cap \mathcal{Z}_k$ . Then either  $\overline{\mathcal{F}} = \mathcal{Z}_k$  or  $\overline{\mathcal{F}}$  is a facet of  $\mathcal{Z}_k$ .*

A formula for the fundamental coweights due to Borel-Tits [BT65, Thm. 6.13] also generalizes almost verbatim. Let  $\omega_\sigma^\vee \in \mathcal{N}_k^1$  be the dual basis of  $\Sigma_k$  such that

$$(10.22) \quad \mathcal{Z}_k = \mathcal{N}_k^0 + \sum_{\sigma \in \Sigma_k} \mathbb{Q}_{\leq 0} \omega_\sigma^\vee.$$

**10.12. Corollary.** *For any non-compact  $\sigma \in \Sigma_K$  the following formula holds:*

$$(10.23) \quad \omega_\sigma^\vee = \sum_{\tau \in \mathcal{G} * \sigma} \omega_\tau^\vee$$

Now we proceed with the construction of a root system. For this, we define

$$(10.24) \quad \Phi_k := W_k \Sigma_k = W_k \operatorname{res}'_A \Sigma_K.$$

generalizing (10.2). One can also consider the restriction of the  $K$ -root system

$$(10.25) \quad \Phi_k^{\operatorname{res}} := \operatorname{res}'_A \Phi_K = \operatorname{res}'_A W_K \Sigma_K$$

but this is, in general, not a root system:

**10.13. Example.** Let  $k = \mathbb{R}$  and  $G = Sp(4, 2)$ . Then  $\operatorname{rk}_{\mathbb{R}} G = 1$  with  $S^0 = \{\alpha_1, \alpha_3\}$ . Let  $H \subseteq G$  be a real form of item 4' of [Was96, Table C]. More precisely, let  $N' \cong \mathbf{G}_a^3$  be the commutator subgroup of  $N$ . Then  $H := MAN' \subseteq G$ . Put  $X = G/H$ . From Wasserman's table we get

$$(10.26) \quad \Sigma_{\mathbb{C}} = \{\sigma_1 := \alpha_1 + \alpha_3, \sigma_2 := \alpha_2\}.$$

Thus,  $\Sigma_{\mathbb{C}}$  generates a root system of type  $G_2$  with the long root  $\sigma_1$  being compact. In particular  $\sigma_1 + n\sigma_2 \in \Phi_{\mathbb{C}}$  for  $n = 1, 2, 3$ . Therefore  $\Sigma_{\mathbb{R}} = \{\overline{\sigma}_2\}$  but

$$(10.27) \quad \Phi_{\mathbb{R}}^{\operatorname{res}} = \{\pm \overline{\sigma}_2, \pm 2\overline{\sigma}_2, \pm 3\overline{\sigma}_2\}$$

which is not a root system.

Nevertheless we have:

**10.14. Theorem.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then*

- i)  $(\Phi_k, \Xi_k)$  is an integral root system. Its Weyl group is  $W_k$  and  $\Sigma_k$  is a system of simple roots.*
- ii)  $n_\sigma \in \{1, 2\}$  for all  $\sigma \in \Sigma_k^{\operatorname{pr}}$  (see Corollary 10.10).*
- iii)  $\Phi_k$  consists precisely of the indivisible elements of  $\Phi_k^{\operatorname{res}}$ , i.e., those  $\sigma \in \Phi_k^{\operatorname{res}}$  such that  $\frac{1}{n}\sigma \notin \Phi_k$  for any  $n \in \mathbb{Z}_{\geq 2}$ .*

*Proof.* *i)* Let  $R_K := \langle \Sigma_K \rangle$  and  $R_k := \operatorname{res}_A R_K = \langle \Sigma_K \rangle$  be the root lattices. Since  $R_K$  is  $W_K$ -stable and therefore  $N(\mathcal{N}_k)$ -stable, we see that  $R_k$  is  $W_k$ -stable. Since the elements of  $\Sigma_k$  are primitive in  $R_k$  by Corollary 10.9, we conclude that  $\Phi_k$  is a root system for  $W_k$  and that  $\Sigma_k$  is a set of simple roots.

It remains to be shown that  $\Xi_k$  is a set of integral weights for  $\Phi_k$ . Since  $\Xi_k$  is  $W_k$ -stable this means that  $W_k$  acts trivially on  $\Xi_k/R_k$ . Clearly, that holds over  $K$ . The assertion now follows from the fact that  $\Xi_k/R_k$  is an quotient of  $\Xi_K/R_K$ .

*ii)* Let  $\sigma \in \Sigma_k^{\text{pr}}$  with  $\tilde{\sigma} = n_\sigma \sigma \in \Sigma_k$ . Since  $\sigma \in \Xi_k$  it follows from *i)* that

$$(10.28) \quad \frac{2}{n_\sigma} = \langle \frac{1}{n_\sigma} \tilde{\sigma}, \tilde{\sigma}^\vee \rangle = \langle \sigma, \tilde{\sigma}^\vee \rangle \in \mathbb{Z}.$$

*iii)* Clearly  $\Phi_k \subseteq \Phi_k^{\text{res}} \subseteq \text{res}_A R_K = R_k$ . Since all elements of  $\Phi_k$  are primitive in  $R_k$  they are all indivisible in  $\Phi_k^{\text{res}}$ . Conversely, let  $\sigma \in \Phi_K$  such that  $\bar{\sigma} = \text{res}_A \sigma \in \Phi_k^{\text{res}}$  is indivisible. Let  $\mathcal{H} = \{\bar{\sigma} = 0\} \subseteq \mathcal{N}_k$  be the hyperplane defined by  $\bar{\sigma}$ , choose  $a \in \mathcal{H}$  off any other reflection hyperplane, and an element  $\bar{w} \in W_k$  with  $\bar{w}a \in \mathcal{Z}_k$ . Then  $\mathcal{F} := \bar{w}\mathcal{H} \cap \mathcal{Z}_k$  is of codimension 1 in  $\mathcal{Z}_k$ . Now lift  $\bar{w}$  to an element  $w \in N(\mathcal{N}_k)$ . Then  $\mathcal{F}$  is the intersection of  $\mathcal{Z}_k$  with  $\mathcal{Z}_K \cap \{w\sigma = 0\}$ . The latter is a face of  $\mathcal{Z}_K$ , hence  $\mathcal{F}$  is a facet of  $\mathcal{Z}_k$ . Let  $\bar{\tau} \in \Sigma_k$  correspond to  $\mathcal{F}$ . Then  $\bar{w}\bar{\sigma}$  is a multiple and therefore equal to  $\bar{\tau}$ . We conclude  $\bar{\sigma} = \bar{w}^{-1}\bar{\tau} \in \Phi_k$ .  $\square$

Summarizing, to any  $K$ -variety  $X$  with  $G$ -action one can associate the following combinatorial objects:

$$(10.29) \quad S, \Xi(T), \Xi_K, \Sigma_K, S^{(p)}$$

If  $X$  is a  $k$ -dense  $G$ -variety one additionally has a subset  $S^0 \subseteq S$  of compact roots and a  $\mathcal{G}^*$ -action on all objects. By restriction to a maximal split  $k$ -torus (computable from the data above by Corollary 10.8) one gets the restricted objects

$$(10.30) \quad S_k, \Xi(A), \Xi_k, \Sigma_k, S_k^{(p)}.$$

We compute the spherical roots in a special case:

**10.15. Proposition.** *Let  $X$  be a  $k$ -dense  $G$ -variety, let  $\mathcal{F}$  be a fan supported in  $\mathcal{Z}_k(X)$ . For any  $\mathcal{C} \in \mathcal{F}$  consider the stratum  $Y := X(\mathcal{C})$  of  $X(\mathcal{F})$ . Then*

$$\begin{aligned} \Sigma_k(Y) &= \Sigma_k(X) \cap \langle \mathcal{C} \rangle^\perp, \\ \Sigma_K(Y) &= \Sigma_K(X) \cap \langle \mathcal{C} \rangle^\perp = \{\sigma \in \Sigma_K(X) \mid \text{res}_A \sigma \in \Sigma_k(Y) \cup \{0\}\}. \end{aligned}$$

*Proof.* Theorem 7.3 *iv)* implies  $\Sigma_k^{\text{pr}}(Y) = \Sigma_k^{\text{pr}}(X) \cap \langle \mathcal{C} \rangle^\perp$ . For  $k = K$  this means  $\Sigma_K(Y) = \Sigma_K(X) \cap \langle \mathcal{C} \rangle^\perp$ . From this we deduce  $\Sigma_k(Y) = \Sigma_k(X) \cap \langle \mathcal{C} \rangle^\perp$  by restriction to  $A$ . Finally let  $\sigma \in \Sigma_K(X)$  and put  $\bar{\sigma} := \text{res}_A \sigma$ . Then  $\sigma \in \Sigma_K(Y)$  if and only if  $\sigma \in \langle \mathcal{C} \rangle^\perp$  if and only if  $\bar{\sigma} \in \langle \mathcal{C} \rangle^\perp$  (since  $\mathcal{C} \subseteq \mathcal{N}(A)$ ) if and only if  $\bar{\sigma} \in \Sigma_k(X) \cup \{0\}$ .  $\square$

From this we get:

**10.16. Corollary.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then*

$$\begin{aligned} \Xi_K(X_{\text{an}}) &= \Xi_K^0(X), \Sigma_K(X_{\text{an}}) = \Sigma_K^0(X), S^{(p)}(X_{\text{an}}) = S^{(p)}(X) \\ \Xi_K(X_{\text{el}}) &= \Xi_K(X), \Sigma_K(X_{\text{el}}) = \Sigma_K^0(X), S^{(p)}(X_{\text{el}}) = S^{(p)}(X) \end{aligned}$$

*Proof.* Let  $\mathcal{C} \subseteq \mathcal{Z}_k(X)$  be a strictly convex cone of maximal dimension. Then  $X(\mathcal{C})$  is induced from  $X_{\text{an}}$  and the data for  $X_{\text{an}}$  follow from Theorem 7.3 and Proposition 10.15. The first equation for  $X_{\text{el}}$  follows from the  $k$ -LST, the two others from the anisotropic case.  $\square$

**10.17. Remark.** Since all orbits in  $X_{\text{an}}$  are affine varieties there are strong constraints on the pair  $(\Sigma_K^0, S^{(p)})$ . For example, if  $X$  is absolutely spherical then  $X_{\text{an}}$  is an affine homogeneous  $K$ -spherical variety. These have been classified by [Krä79], [Mik86], and [Bri87]. See [BP15] for the calculation of the corresponding spherical roots.

The pair  $(S_k, \Sigma_k)$  looks very much like the spherical system of a complex spherical variety. This indeed very often the case but not always.

**10.18. Examples.** *i)* Let  $k = \mathbb{R}$  and  $G = SU(n, n)$  with  $n \geq 2$ . Then the restricted root system of  $G$  is of type  $C_n$  with simple roots  $\beta_i = \bar{\alpha}_i = \bar{\alpha}_{2n-i}$  for  $i = 1, \dots, n$ . Consider the subgroup  $H := U(n, n-1)$  and  $X = G/H$ . Then  $X$  is absolutely spherical of real rank 1 with  $\Sigma_{\mathbb{C}} = \{\sigma\}$  where

$$(10.31) \quad \sigma := \alpha_1 + \dots + \alpha_{2n}.$$

Thus  $\Sigma_{\mathbb{R}} = \{\bar{\sigma}\}$  with

$$(10.32) \quad \bar{\sigma} = 2\beta_1 + \dots + 2\beta_{n-1} + \beta_n.$$

But  $\bar{\sigma}$  is not a spherical root of any spherical variety over  $\mathbb{C}$  (see, e.g., [Was96, Table 1] for a list of possible roots). Observe that rescaling of  $\beta_n$  to  $\frac{1}{2}\beta_n$  converts  $S_{\mathbb{R}}$  to type  $B_n$  and  $\bar{\sigma}$  to the spherical root of  $SO(2n+1)/SO(2n)$ . Therefore, one might wonder whether the normalization of  $S_{\mathbb{R}}$  is chosen correctly. The next example invalidates this objection.

*ii)* Take  $k = \mathbb{R}$  and let  $G$  be the quasi-split group of type  $E_6$  (type  $E_{III}$ ). Its restricted simple roots are

$$(10.33) \quad \beta_1 = \bar{\alpha}_2, \beta_2 = \bar{\alpha}_4, \beta_3 = \bar{\alpha}_3 = \bar{\alpha}_5, \beta_4 = \bar{\alpha}_1 = \bar{\alpha}_6$$

(Bourbaki notation [Bou68]) which span a root system of type  $F_4$ . According to Berger, [Ber57],  $G$  contains a subgroup  $H$  which is isogenous to  $SO(6, 4) \cdot SO(2)$ . Its spherical roots over  $\mathbb{C}$  are

$$(10.34) \quad \sigma_1 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \quad \sigma_2 = \alpha_2 + \alpha_4 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_5$$

(see [BP15]). Their restrictions to  $A$  are

$$(10.35) \quad \bar{\sigma}_1 = \beta_2 + 2\beta_3 + 2\beta_4, \quad \bar{\sigma}_2 = \beta_1 + \beta_2 + \beta_3.$$

They generate a root system of type  $B_2$ . The root  $\bar{\sigma}_1$  is again not a spherical root of any  $\mathbb{C}$ -variety, but worse, Wasserman's tables [Was96] show that  $\bar{\sigma}_1, \bar{\sigma}_2$  are not the spherical roots of any  $F_4$ -variety over  $\mathbb{C}$  even if one allows rescaling the  $\beta_i$  or  $\bar{\sigma}_j$ .

One can also look at the triple

$$(10.36) \quad \Xi_K, \Sigma_K, \Sigma_K^0$$

together with the  $\mathcal{G}^*$ -action, called the *spherical index* of  $X$ , which very much looks and indeed very often is a Satake-Tits diagram. But there are counterexamples: Let  $X = G/H$  be as in Example 10.13. Then  $\Sigma_{\mathbb{C}}$  is of type  $G_2$  and one of the roots is compact. This never occurs in the group case even for any ground field (see [Tit66, p. 61]).

Nevertheless, as in the group case ([BT65, 4.9]) the same major constraint for the structure of indices is still valid:

**10.19. Lemma.** *The spherical index  $(\Xi_K, \Sigma_K, \Sigma_K^0)$  is invariant under the opposition map  $\iota = -w_0$  (where  $w_0 \in W_K$  is the longest element).*

*Proof.* The  $\mathcal{G}^*$ -action stabilizes both  $W_K$  and  $\Sigma_K$ . This implies that  $\mathcal{G}^*$  commutes with  $w_0$  and therefore with  $\iota$ . It remains to be shown that  $\iota\Sigma_K^0 = \Sigma_K^0$ . Let  $\Xi^0 := \text{kern res}_A \subseteq \Xi_K$ . Then  $N(\mathcal{N}_k)$  (the normalizer of  $\mathcal{N}_k$  in  $W_K$ ) acts on  $\Xi^0$ . Let  $\bar{w}_0 \in W_k$  be the longest element and let  $n_0 \in N(\mathcal{N}_k)$  be a lift. Then  $\mathcal{Z}_k = -\bar{w}_0\mathcal{Z}_k \subseteq -n_0\mathcal{Z}_k$ . Let  $\mathcal{F} \subseteq \mathcal{Z}_K$  be the face

corresponding to  $\Sigma_K^0$ . Since  $\mathcal{Z}_k$  contains an interior point of  $\mathcal{F}$  we get  $\mathcal{F} \subseteq \mathcal{Z}_K \cap (-n_0 \mathcal{Z}_K)$ . Thus there is  $w^0 \in W_K^0$  with  $-w^0 n_0 \mathcal{Z}_K = \mathcal{Z}_K$ . By uniqueness we get  $w_0 = w^0 n_0$ . Hence,  
(10.37)  $\iota \Sigma_K^0 = -w^0 n_0 \Sigma_K^0 \subseteq \Sigma_K \cap \Xi^0 = \Sigma_K^0$ .  $\square$

## 11. Wonderful varieties

In this section, we use our results on root systems to investigate smoothness properties of the standard embedding of a  $k$ -convex variety. As a direct consequence of the fact that the valuation cone is simplicial (Corollary 9.7) we get:

**11.1. Theorem.** *Let  $X$  be a  $k$ -convex  $G$ -variety with  $\text{rk}_k X = r$ . Then the standard embedding  $X_{\text{st}}$  contains  $r$  irreducible divisors  $X_1, \dots, X_r$  (namely the codimension-1-strata) such that the strata are exactly the (set theoretic) intersections  $X_I := \bigcap_{i \notin I} X_i$  where  $I$  runs through all subsets of  $\{1, \dots, r\}$ .*

A more canonical way to parametrize strata is by their set of spherical roots:

**11.2. Corollary.** *Let  $X$  be a  $k$ -convex  $G$ -variety and let  $X_{\text{st}}$  be its standard embedding. Then there is a bijection  $J \leftrightarrow X^J$  of subsets  $J$  of  $\Sigma_k(X)$  and strata  $X^J$  of  $X_{\text{st}}$  with*

$$(11.1) \quad \Sigma_k(X^J) = J.$$

*Proof.* Each divisor  $X_i$  corresponds to an extremal ray  $\mathcal{R}_i$  of  $\mathcal{Z}_k(X)$ . Thus there is precisely one  $\sigma_i \in \Sigma_k(X)$  with  $\sigma_i(\mathcal{R}_i) \neq 0$  and

$$(11.2) \quad \Sigma_k(X_I) = \{\sigma_i \mid \sigma_i(\mathcal{R}_j) = 0 \text{ for all } j \in I\} = \{\sigma_i \mid i \notin I\}.$$

Thus, if we set

$$(11.3) \quad J := \{i = 1, \dots, r \mid \sigma_i \notin I\}$$

then  $X^I := X_J$  has the desired property.  $\square$

**11.3. Remark.** One application is that given any subset  $I$  of  $k$ -spherical roots there is a variety, namely  $X^I$ , with that set as a set of spherical roots. The variety  $X^I$  is called. This variety is therefore called the *localization of  $X$  in  $\Sigma$* .

Together with the invariance under the opposition map (10.19), the existence of  $X^I$  puts enormous constraints on the possible subsets  $\Sigma_K^0 \subseteq \Sigma_K$ . For example, if  $\Sigma_K$  is of inner type  $A_n$  then  $\Sigma_K \setminus \Sigma_K^0$  must be of the form  $\{\alpha_d, \alpha_{2d}, \dots, \alpha_{n+1-d}\}$  with  $d \mid n+1$ .

We now address the problem of when the standard embedding is actually smooth. Clearly, this is the case if and only if the torus embedding  $A_k(\mathcal{F}_{\text{st}})$  is smooth. A well known criterion of toroidal geometry asserts that this is the case if and only if the cone  $\mathcal{Z}_k(X)$  is spanned by an integral basis. Dualizing, this leads to:

**11.4. Definition.** A  $k$ -dense  $G$ -variety is called  *$k$ -wonderful* if  $\Sigma_k^{\text{pt}}(X)$  is a  $\mathbb{Z}$ -basis of  $\Xi_k(X)$ .

Observe that a  $k$ -wonderful variety is automatically  $k$ -convex and therefore has a standard embedding. Thus we have:

**11.5. Proposition.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then  $X$  is  $k$ -wonderful if and only if  $X$  has a smooth standard embedding. In this case the intersections of Theorem 11.1 can be taken to be scheme theoretic.*



For reductive groups, the difference between the weight lattice and the root lattice is related to the center of the group. Something similar is valid in general with the rôle of the center being taken over by the group  $\mathfrak{A}_k := \mathfrak{A}_k(X)$  of  $k$ -central automorphisms of  $X$  (Definition 5.1). This group is a subgroup of  $A_k = A_k(X)$  which means that the restriction map  $\Xi_k \rightarrow \Xi(\mathfrak{A}_k)$  is surjective. Let  $\Gamma_k(X)$  be its kernel, i.e.,  $\Gamma_k(X) = \Xi(A_k/\mathfrak{A}_k)$ . In the following discussion we tacitly assume that the orbit space  $X/\mathfrak{A}_k$  exists. This is certainly the case when  $X$  is homogeneous. In general it can be achieved by replacing  $X$  by a suitable dense open subset. Here is the main result of this section:

**11.6. Theorem.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then the quotient  $X/\mathfrak{A}_k(X)$  is  $k$ -wonderful.*

The theorem will be proved simultaneously with Proposition 11.8 below.

**11.7. Remarks.** *i)* A typical example of a non-wonderful  $k$ -variety is  $X = SL(n, \mathbb{C})/SO(n, \mathbb{C})$ ,  $n \geq 3$ , over  $k = K = \mathbb{C}$ . Then  $\Xi_k(X)$  is 2 times the weight lattice of  $SL(n, \mathbb{C})$  which is not a root lattice.

*ii)* Recall that  $\mathfrak{A}_k(X)$  is finite if and only if  $X$  is  $k$ -convex (Corollary 8.9). Thus for any  $k$ -convex variety  $X$  there is finite abelian group  $E$  of automorphisms (namely  $E = \mathfrak{A}_k(X)$ ) such that  $X/E$  is  $k$ -wonderful. With a bit of luck one has  $E(k) = 1$  in which case  $X(k)$  can be considered as a subset of  $(X/E)(k)$ . This is for example the case when  $X$  is a Riemannian symmetric space (over  $k = \mathbb{R}$ ).

*iii)* A very important special case is that of a  $k$ -dense  $G$ -variety  $X$  with  $\mathfrak{A}_k(X) = 1$ . Then  $X$  is  $k$ -wonderful. Since  $\text{Aut}^G G/H = N_G(H)/H$ , the condition  $\mathfrak{A}_k(X) = 1$  is in particular satisfied when  $X = G/H$  is homogeneous with  $H = N_G(H)$  selfnormalizing. Observe that even in the absolutely spherical case there examples with  $\mathfrak{A}_k(X/\mathfrak{A}_k(X)) \neq 1$ . So Theorem 11.6 cannot be reduced to the case  $\mathfrak{A}_k = 1$ .

Put

$$(11.4) \quad \Sigma_k^{\text{aut}}(X) := \Sigma_k^{\text{pr}}(X/\mathfrak{A}_k).$$

Then another way to phrase the theorem is that  $\Sigma_k^{\text{aut}}$  is a  $\mathbb{Z}$ -basis of  $\Gamma_k = \Xi_k(X/\mathfrak{A}_k)$ . Moreover  $\Sigma_k^{\text{aut}}$  is a system of simple roots for the root system

$$(11.5) \quad \Phi_k^{\text{aut}}(X) := W_k \Sigma_k^{\text{aut}}(X) = \Phi_k^{\text{pr}}(X/\mathfrak{A}_k).$$

with  $\mathcal{Z}_k(X)$  as antidominant Weyl chamber and  $W_k(X)$  as Weyl group. We record some more properties:

**11.8. Proposition.** *Let  $X$  be a  $k$ -dense  $G$ -variety. Then:*

- i)*  $\Xi_k(X)$  is a weight lattice for  $\Phi_k^{\text{aut}}(X)$ . In particular, there is a map  $\Sigma_k^{\text{pr}}(X) \rightarrow \{1, 2\} : \sigma \mapsto n_\sigma^{\text{aut}}$  with  $\Sigma_k^{\text{aut}} = \{n_\sigma^{\text{aut}} \sigma \mid \sigma \in \Sigma_k^{\text{pr}}(X)\}$ .
- ii)*  $\Gamma_k(X) = \text{res}_A \Gamma_K(X)$  and  $\Sigma_k^{\text{aut}} = \text{res}'_A \Sigma_K^{\text{aut}}$ .
- iii)* If  $X$  is  $K$ -wonderful then  $X$  is also  $k$ -wonderful.

*Proof.* For  $k = K$ , Theorem 11.6 is the main result of [Kno96]. From that we derive the general case.

Let  $X_k := X/\mathfrak{A}_k(X)$ . It is easy to see that the group  $\mathfrak{A}_K(X)$  and its action on  $X$  are defined over  $k$ . Therefore also the variety  $X_K := X/\mathfrak{A}_K(X)$  is defined over  $k$ . Because of  $\mathfrak{A}_k \subseteq \mathfrak{A}_K$  these spaces are connected by surjective morphisms

$$(11.6) \quad X \twoheadrightarrow X_k \twoheadrightarrow X_K.$$

We have by definition  $\Xi_k(X_k) = \Gamma_k$  and  $\Xi_K(X_K) = \Gamma_K$ . The equality  $\mathfrak{A}_k = \mathfrak{A}_K \cap A_k$  (eqn. (8.5)) implies that  $A_k/\mathfrak{A}_k \rightarrow A_K/\mathfrak{A}_K$  is injective. Passing to character groups this means that

$$(11.7) \quad \text{res}_A : \Gamma_K \rightarrow \Gamma_k$$

is surjective which already shows the first half of *ii*). From

$$(11.8) \quad \Gamma_K = \Xi_K(X_K) \twoheadrightarrow \Xi_k(X_K) \subseteq \Xi_k(X_k) = \Gamma_k$$

we get  $\Xi_k(X_K) = \Gamma_k$ . Now let

$$(11.9) \quad \Sigma'_k := \text{res}'_A \Sigma_K^{\text{aut}}(X) = \text{res}'_A \Sigma_K(X_K) = \Sigma_k(X_K)$$

which is a linearly independent subset of  $\Gamma_k$ . The main result of [Kno96], Cor. 6.5, asserts that  $\Sigma_K(X_K)$  is a  $\mathbb{Z}$ -basis of  $\Gamma_K$ . It follows that  $\Sigma'_k$  generates  $\Gamma_k$ . Thus it also a  $\mathbb{Z}$ -basis of  $\Gamma_k$  which therefore must equal  $\Sigma_k^{\text{pr}}(X_k) = \Sigma_k^{\text{aut}}(X)$ . This proves Theorem 11.6 and the second half of *ii*).

An even simpler argument proves *iii*): If  $X$  is  $K$ -wonderful then  $\Sigma_K(X)$  is a  $\mathbb{Z}$ -basis of  $\Xi_K(X)$ . Restriction to  $A$  shows that  $\Sigma_k(X)$  generates and therefore is a  $\mathbb{Z}$ -basis of  $\Xi_k(X)$ . Since then  $\Sigma_k(X) = \Sigma_k^{\text{pr}}(X)$ , it follows that  $X$  is  $k$ -wonderful, as well.

Finally, it is known that  $W_K$  acts trivially on  $\Xi_K/\Gamma_K$  (by [Kno96, Cor. 6.5c]). It follows that  $W_k$  acts trivially on the quotient  $\Xi_k/\Gamma_k$ . This means that  $\Xi_k$  is a weight lattice for  $\Phi_k^{\text{aut}}$  and proves *i*).  $\square$

**11.9. Remark.** All elements of  $\Sigma_K^{\text{aut}}(X)$  are integral multiples of elements of  $\Sigma_K(X) = \Sigma_K^{\text{pr}}(X)$ . It follows by restriction that all elements of  $\Sigma_k^{\text{aut}}(X)$  are integral multiples of elements of  $\Sigma_k(X)$ . This implies  $n_\sigma \leq n_\sigma^{\text{aut}}$  where  $n_\sigma$  was defined in Corollary 10.10

## 12. Boundary degenerations

Let, for simplicity,  $X = G/H$  be a homogeneous  $k$ -spherical variety with character group  $\Xi = \Xi_k(X)$  and let  $\mathfrak{A} := \mathfrak{A}_k(X) \subseteq A_k = A_k(X)$  be its group of central automorphisms. Then  $\Xi(A_k/\mathfrak{A}) = \mathbb{Z}\Sigma$  where  $\Sigma = \Sigma_k^{\text{aut}}(X)$ .

The group  $\mathfrak{A}$  acts freely on  $X$ . The quotient  $Y := X/\mathfrak{A}$  has the property that  $\Sigma_k(Y) = \Sigma$  and  $\Xi_k(Y) = \mathbb{Z}\Sigma$ . Thus,  $Y$  admits a smooth standard embedding  $\mathfrak{Y} := Y_{\text{st}}$ .

Now put  $Z := (X \times X)/\mathfrak{A}$  where  $\mathfrak{A}$  acts diagonally. Then  $Z$  is a  $k$ -spherical  $G \times G$ -variety with

$$(12.1) \quad \Xi_k(Z) = \{(\chi, \eta) \in \Xi \oplus \Xi \mid \chi + \eta \in \mathbb{Z}\Sigma\}.$$

Thus there is a short exact sequence

$$(12.2) \quad 0 \rightarrow \Xi \xrightarrow{\Delta_-} \Xi_k(Z) \xrightarrow{\beta} \mathbb{Z}\Sigma \rightarrow 0$$

with  $\Delta_-(\chi) = (\chi, -\chi)$  and  $\beta(\chi, \eta) = \chi + \eta$ . In particular,  $\beta$  induces an injection

$$(12.3) \quad \beta^* : \mathcal{N}_k(Y) \hookrightarrow \mathcal{N}_k(Z).$$

Since  $\Sigma_k(Z) = (\Sigma \times 0) \cup (0 \times \Sigma)$  we have  $\beta(\Sigma_k(Z)) = \Sigma$  and therefore

$$(12.4) \quad \mathcal{C}_{\text{bd}} := \beta^*(\mathcal{Z}_k(Y)) \subseteq \mathcal{Z}_k(Z).$$

Let  $\mathcal{F}_{\text{bd}}$  be the fan of faces of  $\mathcal{C}_{\text{bd}}$  and let  $\mathfrak{X} := Z(\mathcal{F}_{\text{bd}})$  be the corresponding embedding of  $Z$ .

**12.1. Theorem.** *Let  $X$  be a homogeneous  $k$ -spherical variety.*

*i) There are unique morphisms*

$$(12.5) \quad p : \mathfrak{X} \rightarrow \mathfrak{Y}, \quad \delta : \mathfrak{Y} \rightarrow \mathfrak{X}$$

*extending the first projection  $Z \rightarrow Y : (x_1, x_2)/\mathfrak{A} \mapsto x_1/\mathfrak{A}$  and the diagonal morphism  $Y \hookrightarrow Z : x/\mathfrak{A} \mapsto (x, x)/\mathfrak{A}$ , respectively.*

*ii) The spaces  $\mathfrak{X}$ ,  $\mathfrak{Y}$  and the morphism  $p$  are smooth while  $\delta$  is a section of  $p$ . In particular,  $\delta$  is a closed embedding and  $p$  is surjective.*

*iii) For any  $y \in \mathfrak{Y}(k)$  the fiber  $\mathfrak{X}_y := p^{-1}(y)$  is a homogeneous  $k$ -spherical  $G$ -variety with base point  $\delta(y)$ . Its spherical data are*

$$(12.6) \quad \Xi_k(\mathfrak{X}_y) = \Xi_k(X), \quad S^{(p)}(\mathfrak{X}_y) = S^{(p)}(X), \quad \Sigma_K(\mathfrak{X}_y) = \Sigma_K(Gy).$$

*Moreover,  $\mathfrak{X}_y$  is a  $k$ -form of  $X$  or horospherical if and only if  $Gy$  is the open or the closed orbit of  $\mathfrak{Y}$ , respectively.*

*iv) Let  $\mathcal{F}$  be a fan supported in  $\mathcal{Z}_k(X)$  and  $\Delta\mathcal{F} \subseteq \mathcal{Z}_k(X) \times \mathcal{Z}_k(X) = \mathcal{Z}_k(X \times X)$  the diagonal fan. Then there is a Cartesian diagram*

$$(12.7) \quad \begin{array}{ccc} (X \times X)(\Delta\mathcal{F}) & \longrightarrow & \mathfrak{X} \\ \downarrow q & & \downarrow p \\ X(\mathcal{F}) & \xrightarrow{\pi} & \mathfrak{Y} \end{array}$$

*where  $\pi$  and  $q$  extend the morphism  $X \rightarrow Y$  and the first projection  $X \times X \rightarrow X$ , respectively.*

*v) For any  $\mathcal{C} \in \mathcal{F}$  let  $x \in X(\mathcal{C})(k)$ . Put  $y := \varphi(x) \in \mathfrak{Y}$  and  $z := \delta(y) \in \mathfrak{X}_y$ . Then there is a short exact sequence*

$$(12.8) \quad 1 \rightarrow G_z \rightarrow G_x \rightarrow A_0 \rightarrow 1$$

*where  $A_0$  is the connected subgroup of  $A_k(X)$  with  $\mathcal{N}(A_0) = \langle \mathcal{C} \rangle_{\mathbb{Q}} \subseteq \mathcal{N}_k(X)$ .*

*Proof.* *i)* The projection  $p_1 : Z \rightarrow Y$  induces the map

$$(12.9) \quad \mathcal{N}(p_1) : \mathcal{N}_k(Z) \rightarrow \mathcal{N}_k(Y)$$

of which  $\beta^*$  is a section. Thus  $\mathcal{N}(p)$  maps  $\mathcal{C}_{\text{bd}}$  isomorphically onto  $\mathcal{Z}_k(Y)$  which shows that  $p_1$  extends to a morphism  $p$  (Corollary 7.8). Using the local structure theorem, the extendability of the diagonal map is reduced to that between torus embeddings

$$(12.10) \quad (A/\mathfrak{A})(\mathcal{F}_{\text{st}}) \dashrightarrow ((A \times A)/\mathfrak{A})(\mathcal{F}_{\text{bd}})$$

which is clear by construction. Then  $p \circ \delta = \text{id}_{\mathfrak{Y}}$  follows by continuity.

*ii)* We already know that  $\mathfrak{Y}$  is smooth. Since  $\beta$  has a section also  $\mathcal{C}_{\text{bd}}$  is spanned by a partial basis of  $\text{Hom}(\Xi_k(Z), \mathbb{Z})$  which means that  $\mathfrak{X}$  is smooth, as well.

Clearly  $p$  is smooth in all image points of the section  $\delta$ . Now, the fact that  $\mathcal{C}_{\text{bd}} \rightarrow \mathcal{Z}_k(Y)$  is bijective means that  $p$  induces a bijection between the set of  $G \times G$ -orbit in  $\mathfrak{X}$  and

$G \times 1$ -orbits in  $\mathfrak{Y}$ . This implies that  $1 \times G$  acts transitively on all fibers of  $p$ . This shows that  $p$  is smooth everywhere.

*iii)* We already know that  $\mathfrak{X}_y$  is  $G$ -homogeneous with base point  $\delta(p)$ . Assume that  $Gy$  is the closed orbit of  $\mathfrak{Y}$ . Then  $p^{-1}(Gy)$  is the closed orbit of  $\mathfrak{X}$  which is horospherical since  $\mathcal{C}_{\text{bd}}$  meets the interior of  $\mathcal{Z}_k(Z)$ . Thus there exists an  $N \times N$  fixed point  $z \in p^{-1}(Gy)$ . But then  $z$  is an  $N$ -fixed point of the fiber  $\mathfrak{X}_{p(z)}$  and is therefore horospherical, as well. Being also homogeneous this means that  $\mathfrak{X}_{p(z)}$  is spherical which means that  $P$  has dense orbit. That being an open condition we conclude that  $\mathfrak{X}_y$  is spherical for  $y$  in an open neighborhood of  $p(z)$ . By  $G \times 1$ -equivariance we conclude that  $\mathfrak{X}_y$  is  $k$ -spherical for all  $y \in \mathfrak{Y}(k)$ . The calculation of the spherical data is deferred to part *v*).

*iv)* First consider the diagram of open subsets:

$$(12.11) \quad \begin{array}{ccc} X \times X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where the vertical maps are projections onto the first factor. Because  $X \rightarrow Y$  is a principal  $\mathfrak{A}$ -bundle we have  $X \times_Y X \cong X \times \mathfrak{A}$ . Then

$$(12.12) \quad X \times_Y Z = (X \times_Y X \times X)/\mathfrak{A} \cong (X \times \mathfrak{A} \times X)/\mathfrak{A} = X \times X$$

shows that the diagram (12.11) is Cartesian. Next we show that the maps in (12.11) extend to embeddings. For this it suffices (again by Corollary 7.8) that the corresponding maps of  $\mathcal{N}$ -spaces

$$(12.13) \quad \begin{array}{ccc} \mathcal{N}(X) \times \mathcal{N}(X) & \longrightarrow & (\mathcal{N}(X) \times \mathcal{N}(X))/\mathcal{N}(\mathfrak{A}) \\ \downarrow & & \downarrow \\ \mathcal{N}(X) & \longrightarrow & \mathcal{N}(X)/\mathcal{N}(\mathfrak{A}) \end{array}$$

map fans to each other. But that is clear since  $\mathcal{C}_{\text{bd}}$  is nothing else than the image of

$$(12.14) \quad \mathcal{Z}_k(Y) \hookrightarrow \mathcal{N}(X)/\mathcal{N}(\mathfrak{A}) \xrightarrow{\text{diag}} (\mathcal{N}(X) \times \mathcal{N}(X))/\mathcal{N}(\mathfrak{A}).$$

It remains to show that

$$(12.15) \quad u : (X \times X)(\mathcal{F}) \rightarrow V := X(\mathcal{F}) \times_{\mathfrak{Y}} \mathfrak{X}$$

is an isomorphism. An argument as above shows that the fibers of  $q$  are the  $1 \times G$ -orbits. Semicontinuity implies that all fibers of  $q$  are irreducible of the same dimension. This shows that  $u$  is birational, surjective and quasi-finite. The target  $V$  is smooth over  $X(\mathcal{F})$  hence normal. Zariski's main theorem implies that  $u$  is an open embedding hence an isomorphism.

*v)* Consider first the map of  $G \times G$ -orbits

$$(12.16) \quad Z_1 := (X \times X)(\Delta\mathcal{C}) \rightarrow Z_2 := (X \times X)(\mathcal{C} \times \mathcal{C}) = X(\mathcal{C}) \times X(\mathcal{C}).$$

It follows from the local structure theorem and toroidal theory that this is a principal bundle with a structure group  $A_0$  with  $\Xi(A_0) = \Delta\mathcal{C}^\perp \cap \Xi^2/(\mathcal{C}^\perp \times \mathcal{C}^\perp) \cap \Xi^2$ . Projection to the first component yields an isomorphism  $\Xi(A_0) = \text{res}_{\langle \mathcal{C} \rangle} \Xi$ . Thus,  $A_0 \subseteq A_k$  is connected with  $\mathcal{N}(A_0) = \langle \mathcal{C} \rangle$ . For isotropy group this means

$$(12.17) \quad 1 \rightarrow (G \times G)_{(x,z)} \rightarrow G_x \times G_x \xrightarrow{\eta_1, \eta} A_0 \rightarrow 1.$$

We have  $\eta_1 = \eta_2$  by symmetry which implies that  $\eta_1$  is surjective. Then  $G_z = \ker \eta_1$  and the assertion follows.

Finally we compute the spherical data of  $\mathfrak{X}_y = G/G_z$ . Then clearly  $S^{(p)}(G/G_z) = S^{(p)}(G/G_x) = S^{(p)}(X(\mathcal{C})) = S^{(p)}(X(\mathcal{C}))$ . The same argument yields  $\Sigma_k(G/G_z) = \Sigma_k(G/G_x) = \Sigma(y)$ . Finally  $\Xi(\mathfrak{X}_y) = \Xi$  since every  $AN$ -semiinvariant rational function can be extended to  $\mathfrak{X}$ .

It follows that  $\mathfrak{X}_y$  is horospherical if and only if  $\Sigma(y) = \emptyset$  if and only if  $Gy$  is closed in  $\mathfrak{Y}$ . On the other hand, for any  $y \in Y$  let  $u \in X(K)$  be a lift. Then  $X \rightarrow \mathfrak{X}_y : x/\mathfrak{A} \mapsto (u, x)/\mathfrak{A}$  is a  $K$ -isomorphism.  $\square$

**12.2. Remark.** The varieties  $\mathfrak{X}_y$  are called the *boundary degenerations of  $X$* . The last point *v)* gives another interpretation: it means that the isotropy groups  $G_{x'} \subseteq G$  have a well-defined limit  $G_z \subseteq G$  whenever  $x' \in X$  tends to some point  $x \in X(\mathcal{F})$ . Moreover, this limit subgroup is a normal subgroup of  $G_x$ , the quotient being the split torus  $A_0$ . Moreover For that reason, the spaces  $G/G_{x'} = \mathfrak{X}_{\pi(x)}$  is a boundary degenerations of  $X$ .

The fibration  $\mathfrak{X} \rightarrow \mathfrak{Y}$  could be called the *big boundary degeneration of  $X$*  since because of the  $G$ -action on the base many fibers  $\mathfrak{X}_y$  are isomorphic to each others. It is possible to cut down on the amount of isomorphic fibers by restricting the base to a subspace. For this, let  $\tilde{A} := A_k/\mathfrak{A} = A_k(\mathfrak{Y})$  and  $\tilde{\mathfrak{Y}} \subseteq \mathfrak{Y}_{\text{el}} \subseteq \mathfrak{Y}$  the affine closure of a flat. Now define  $\tilde{\mathfrak{X}} := \mathfrak{X} \times_{\mathfrak{Y}} \tilde{\mathfrak{Y}}$ . Then

$$(12.18) \quad \tilde{p} : \tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{Y}}$$

is also a smooth deformation of  $X$  which we call the *small boundary degeneration of  $X$* . There is an action of  $G \times \tilde{A}$  on  $\tilde{\mathfrak{X}}$  for which  $\tilde{p}$  is equivariant.

Next we give an infinitesimal version of the boundary degeneration. For this let  $\text{Grass}(\mathfrak{g})$  be the Grassmann variety of linear subspaces of  $\text{Lie } G$ . Then for every homogeneous  $G$ -variety  $X$  there is a canonical morphism  $\Phi : X \rightarrow \text{Grass}(\mathfrak{g})$  mapping  $x \in X$  to the isotropy subalgebra  $\mathfrak{g}_x$ . If  $x \in X(k)$  is a base point with  $\mathfrak{h} := \mathfrak{g}_x$  then  $\Phi$  is just the orbit map  $\Phi(gx) = \text{Ad } g \mathfrak{h}$ .

**12.3. Corollary.** *Let  $X$  be a homogeneous  $k$ -spherical variety and let  $\mathcal{F}$  be a fan supported in  $\mathcal{Z}_k(X)$ .*

*i) The morphism  $\Phi : X \rightarrow \text{Grass}(\mathfrak{g})$  extends uniquely to a morphism*

$$(12.19) \quad \bar{\Phi} : X(\mathcal{F}) \rightarrow \text{Grass}(\mathfrak{g}).$$

*Moreover, this  $\bar{\Phi}$  factors uniquely through  $\mathfrak{Y}$ , i.e., there is a  $G$ -morphism  $\Phi' : \mathfrak{Y} \rightarrow \text{Grass}(\mathfrak{g})$  which is independent of  $\mathcal{F}$  such that  $\bar{\Phi} = \Phi' \circ \pi$ :*

$$(12.20) \quad \begin{array}{ccc} X \hookrightarrow X(\mathcal{F}) & & \\ \downarrow & \searrow \bar{\Phi} & \\ Y \hookrightarrow \mathfrak{Y} & \xrightarrow{\Phi'} & \text{Grass}(\mathfrak{g}). \end{array}$$

*ii) For  $\mathcal{C} \in \mathcal{F}$  let  $\mathfrak{a}_0 := \langle \mathcal{C} \rangle_{\mathbb{Q}} \otimes_{\mathbb{Q}} k \subseteq \mathcal{N}_k(X) \otimes_{\mathbb{Q}} k = \text{Lie } A_k(X)$  and let  $x \in X(\mathcal{C})$ . Then there is a short exact sequence of Lie algebras*

$$(12.21) \quad 0 \rightarrow \bar{\Phi}(x) \rightarrow \mathfrak{g}_x \rightarrow \mathfrak{a}(\mathcal{C}) \rightarrow 0.$$

*iii) Let  $F = \overline{Ay} \subseteq \mathfrak{Y}$  be a flat. Then the restriction of  $\Phi'$  to  $F$  is a finite morphism.*

*Proof.* *i)* The morphism  $\Phi'$  is simply  $y \mapsto \mathfrak{g}_{\delta(y)}$ . Then  $\overline{\Phi} := \Phi' \circ \pi$  is clearly an extension of  $\Phi$ .

*ii)* This follows from part *v)* of Theorem 12.1.

*iii)* From  $N_G(H^0) = N_G(\mathfrak{h})$  and  $\mathcal{Z}_k(G/H) = \mathcal{Z}_k(G/H^0)$  it follows that  $\mathfrak{A}^0$  is also the maximal split subtorus of  $N_G(\mathfrak{h})/H^0$ . This shows that  $N_A(\mathfrak{h})^0 \subseteq \mathfrak{A}H = G_y$  and therefore that the restriction of  $\Phi'$  to  $Ay$  is quasi-finite. Now let  $F \xrightarrow{\varphi} \tilde{F} \xrightarrow{\psi} \text{Grass}(\mathfrak{g})$  be the Stein factorization, i.e.,  $\varphi$  has connected and  $\psi$  has finite fibers. We have seen above that  $\varphi$  is in fact birational and we have to show that it is an isomorphism. The fan corresponding to the toroidal embedding  $F$  is  $\mathcal{F} := W_k \mathcal{F}_{\text{st}}(Y)$ . Let  $\tilde{\mathcal{F}}$  be the fan corresponding to  $\tilde{F}$ . Suppose they are not equal. Then there is a maximal cone  $\mathcal{C} \in \mathcal{F}$  with  $\mathcal{C} \notin \tilde{\mathcal{F}}$ . By  $W_k$ -symmetry we may assume that  $\mathcal{C} = \mathcal{Z}_k(Y)$ . Since  $\mathcal{F}$  is a refinement of  $\tilde{\mathcal{F}}$  there is a  $\tilde{\mathcal{C}} \in \tilde{\mathcal{F}}$  with  $\mathcal{C} \subsetneq \tilde{\mathcal{C}}$ . Thus  $\mathcal{C}$  has a facet  $\mathcal{C}_0$ , corresponding to a root  $\sigma \in \Sigma_k(Y)$ , which is not a facet of  $\tilde{\mathcal{C}}$ . It follows that the  $A_k(Y)$ -orbits corresponding to  $\mathcal{C}$  and  $\mathcal{C}_0$  are mapped to the same orbit in  $\tilde{F}$ . But that is not possible since their images in  $\text{Grass}(\mathfrak{g})$  lie in different  $G$ -orbits: since the boundary degenerations have different sets of spherical roots (namely  $\emptyset$  and  $\{\sigma\}$ , respectively) they are not even isomorphic as  $G$ -varieties. This completes the proof that  $F \rightarrow \tilde{F}$  is an isomorphism.  $\square$

**12.4. Remark.** When  $k$  is algebraically closed it is known that  $\Phi'$  is a closed embedding (see [Los09] for the last step) which means that  $\overline{\Phi}$  cannot be factored any further. This is not true in general since  $\mathfrak{A}_k(X)$  may not constitute the full automorphism group of  $X$ . Nevertheless, it might be possible that the restriction of  $\Phi'$  to a flat  $F$  is always a closed embedding.

Following [Bri90], one can use the morphism  $\Phi'$  to construct a distinguished basis for  $\mathfrak{h}$ . More precisely, let  $X = G/H$  be a homogeneous  $k$ -spherical variety and let  $Q = LU$  be an adapted parabolic such that  $x_0 := eH \in X_{\text{el}}$  and let  $y_0$  be its image in  $Y$ . Let  $L_0 := L_{x_0} = L \cap H$  such that  $X_{\text{el}} = L/L_0$ . Because of  $\mathfrak{l} = \mathfrak{l}_0 + \mathfrak{m} + \mathfrak{a}$  there is an  $AL_0$ -stable subspace  $\tilde{\mathfrak{l}} \subseteq \mathfrak{m} \oplus \mathfrak{a}$  such that  $\mathfrak{l} = \mathfrak{l}_0 \oplus \tilde{\mathfrak{l}}$  (whence  $\mathfrak{a} \cap \tilde{\mathfrak{l}} \cong \mathfrak{a}_k(X)$ ). We also choose an  $L_0$ -stable splitting  $\mathfrak{h} = \mathfrak{l}_0 \oplus \tilde{\mathfrak{h}}$ . Then the  $k$ -LST implies

$$(12.22) \quad \mathfrak{g} = \mathfrak{u} \oplus \tilde{\mathfrak{l}} \oplus \mathfrak{l}_0 \oplus \tilde{\mathfrak{h}}$$

Now let  $y_0 \in Y$  be the image of  $x_0$ , let  $y_1 \in \mathfrak{Y}$  be the fixed point in  $\overline{Ay_0} \subseteq \mathfrak{Y}_{\text{el}}$ , and let  $x_1 := \delta(y_1)$ . Then  $\mathfrak{g}_{x_1} = \mathfrak{l}_0 \oplus \mathfrak{u}^-$  and therefore

$$(12.23) \quad \mathfrak{g} = \mathfrak{u} \oplus \tilde{\mathfrak{l}} \oplus \mathfrak{l}_0 \oplus \mathfrak{u}^-$$

This shows that  $\tilde{\mathfrak{h}}$  is the graph of an  $L_0$ -equivariant homomorphism  $\Upsilon : \mathfrak{u}^- \rightarrow \mathfrak{u} \oplus \mathfrak{l}$  inside  $\mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{u}^-$  which implies

$$(12.24) \quad \mathfrak{h} = \mathfrak{l}_0 \oplus \{\Upsilon(\xi) + \xi \mid \xi \in \mathfrak{u}^-\}.$$

To make this more precise, let  $\Phi_{\mathfrak{u}}$  be the set of restricted roots  $\alpha \in \Phi_k$  with  $\mathfrak{g}_{\alpha} \subseteq \mathfrak{u}$ . Then for all  $\alpha, \beta \in \Phi_{\mathfrak{u}}$  consider the component

$$(12.25) \quad \Upsilon_{\alpha\beta} : \mathfrak{g}_{-\alpha} \hookrightarrow \mathfrak{u}^- \xrightarrow{\Upsilon} \mathfrak{u} \oplus \tilde{\mathfrak{l}} \rightarrow \mathfrak{g}_{\beta}$$

For  $\beta = 0$  we put

$$(12.26) \quad \Upsilon_{\alpha 0} : \mathfrak{g}_{-\alpha} \hookrightarrow \mathfrak{u}^- \xrightarrow{\Upsilon} \mathfrak{u} \oplus \tilde{\mathfrak{l}} \rightarrow \tilde{\mathfrak{l}}.$$

The follow statement can be seen as a vanishing theorem for components:

### 12.5. Corollary. *Let*

$$(12.27) \quad \mathcal{M} := \{\alpha + \beta \mid \alpha \in \Phi_{\mathfrak{u}}, \beta \in \Phi_{\mathfrak{u}} \cup \{0\}, \text{ and } \Upsilon_{\alpha\beta} \neq 0\}.$$

Then  $\mathcal{M}$  and  $\Sigma_k(X)$  generate the same convex cone in  $\Xi(A) \otimes \mathbb{Q}$ . In particular,  $\mathcal{M} \subseteq \Xi_k(X) \otimes \mathbb{Q}$  and

$$(12.28) \quad \mathcal{Z}_k(X) = \{a \in \mathcal{N}_k(X) \mid \sigma(a) \leq 0 \text{ for all } \sigma \in \mathcal{M}\}.$$

*Proof.* Consider the closure  $\overline{V}$  of  $\text{Ad } A(\mathfrak{h})$  in  $\text{Grass}(\mathfrak{g})$  it contains  $\mathfrak{g}_{x_1}$  as a fixed point. Let  $V \subseteq \overline{V}$  be the  $A$ -stable open affine neighborhood of this fixed point. Its normalization is the affine toroidal embedding corresponding to the cone  $\mathcal{C}$  which is dual to  $\mathbb{Q}_{\geq 0}\mathcal{M}$  (see the proof of [Bri90, Thm. 2.3]). On the other hand, this cone equals  $\mathcal{Z}_k(Y)$  according to Corollary 12.3 iii). Thus, the dual cones are the same, as well.  $\square$

## 13. The weak polar decomposition

We present an application of our theory to local fields.

**13.1. Proposition.** *Let  $k$  be a local field and let  $X$  be a homogeneous  $G$ -variety of rank 0. Then  $G(k)$  acts transitively on  $X(k)$ .*

*Proof.* Let  $X = G/H$ . Then, by Corollary 8.5, there is a  $k$ -parabolic  $Q = LU \subseteq G$  with  $Q_{\text{an}} \subseteq H \subseteq Q$ . Let  $G_0$  be the largest anisotropic factor of  $L$ . Then  $Q = G_0Q_{\text{an}}$  and therefore the action of  $G_0$  on  $X_0 := Q/H$  is transitive. Since  $G(k)$  acts transitively on  $(G/Q)(k)$  ([BT65, Thm. 4.13 a)]) it suffices to show that  $G_0(k)$  acts transitively on  $X_0(k)$ . Thus, replacing  $(G, X)$  by  $(G_0, X_0)$  we are reduced to the case that  $G$  is anisotropic.

Then  $G(k)$  and  $X(k)$  are compact. Now suppose  $G(k)$  does not act transitively on  $X$  and let  $X_1 \subseteq X(k)$  be an orbit. Since  $G(k)$  has only finitely many orbits (Corollary 4.15) and all of them are closed the complement  $X_2 = X(k) \setminus X_1$  is closed and nonempty. The variety  $X$  is also an affine variety (see Proposition 3.4). Therefore, the Stone-Weierstraß theorem holds for regular functions on  $X(k)$  (see [Die44, Kap50]). Thus, approximating the characteristic function of  $X_1$  we obtain  $f \in k[X]$  with  $|f(x) - 1| < \frac{1}{2}$  for  $x \in X_1$  and  $|f(x)| < \frac{1}{2}$  for  $x \in X_2$ . Let  $\overline{f}(x) := \int_{G(k)} f(gx) dg$ . Then still  $|\overline{f}(x) - 1| < \frac{1}{2}$  for  $x \in X_1$  and  $|\overline{f}(x)| < \frac{1}{2}$  for  $x \in X_2$  which implies that  $\overline{f}$  is not constant. On the other hand, since  $G(k)$  is Zariski-dense in  $G$  and  $\overline{f}$  is  $G(k)$ -invariant it is also  $G$ -invariant. Thus it must be constant since  $G$  acts transitively on  $X$ .  $\square$

We generalize this to  $k$ -spherical varieties of arbitrary rank. Let  $|\cdot|$  be an absolute value on the local field  $k$ . Let  $X$  be a  $k$ -dense  $G$ -variety and  $A_k := A_k(X)$ . Then we call

$$(13.1) \quad A_k^+ := \{x \in A_k(k) \mid |\sigma(x)| \leq 1 \text{ for all } \sigma \in \Sigma_k(X)\}$$

the *compression domain* in  $A_k$ . Then the following weak form of a polar decomposition holds.

**13.2. Theorem.** *Let  $k$  be a local field, let  $X$  be a homogeneous  $k$ -spherical variety and let  $x_0 \in X(k)$  such that  $A_k x_0 \subseteq X$  is a flat. Then there is a compact subset  $\Omega \subseteq G(k)$  with*

$$(13.2) \quad X(k) = \Omega \cdot A_k^+ x_0.$$

*Proof.* Let  $\mathcal{F}$  be a complete fan  $\mathcal{Z}_k(X)$  which we also choose to be smooth, i.e., each cone in  $\mathcal{F}$  is generated by a partial integral basis. Let  $\overline{A}_k := A(\mathcal{F})$  and  $\overline{X} := X(\mathcal{F})$ . Then both  $\overline{A}_k$  and  $\overline{X}$  are smooth.

Let  $\mathcal{C} \in \mathcal{F}$  be of maximal dimension. Then the corresponding orbit  $A_k(\mathcal{C})$  consists of a single fixed point  $a_{\mathcal{C}}$ . Let  $\mathcal{C}$  be defined by inequalities  $\beta_1 \leq 0, \dots, \beta_r \leq 0$ . If we assume that the  $\beta_i$  are primitive in  $\Xi_k(X)$  they even form an integral basis (by smoothness). A neighborhood of  $a_{\mathcal{C}}$  in  $\overline{A}_k(\mathcal{F})$  is isomorphic to the affine space  $\mathbf{A}^r$  where  $A_k$  acts with the characters  $\beta_i$ . Thus, if we put

$$(13.3) \quad A_{\mathcal{C}}^+ := \{x \in A_k(k) \mid |\beta_i(x)| \leq 1 \text{ for all } i = 1, \dots, r\}$$

then its closure  $\overline{A}_{\mathcal{C}}^+$  in  $\overline{A}(k)$  contains  $D^r \subseteq \mathbf{A}^r(k)$  where  $D \subseteq k$  is the unit disk. This implies that  $\overline{A}_{\mathcal{C}}^+$  is a neighborhood of  $a_{\mathcal{C}}$  in  $\overline{A}$ .

Since the  $\beta_i$  form an integral basis of  $\Xi_k(X)$  one can express the spherical roots in terms of the  $\beta_i$ :

$$(13.4) \quad \sigma = \sum_{i=1}^r \nu_i(\sigma) \beta_i \text{ with } \nu_i(\sigma) \in \mathbb{Z}.$$

From the inclusion  $\mathcal{C} \subseteq \mathcal{Z}_k(X)$  we get  $\nu_i(\sigma) \geq 0$  and therefore  $A_{\mathcal{C}}^+ \subseteq A_k^+$ . Thus we have proved: the closure  $\overline{A}_{\mathcal{C}}^+$  of  $A_{\mathcal{C}}^+$  in  $\overline{A}_k(k)$  is a neighborhood of every fixed point  $a_{\mathcal{C}}$ .

Let  $\varphi : \overline{A}_k \rightarrow \overline{X}$  be the embedding extending the orbit map  $a \rightarrow ax_0$ . Then  $x_{\mathcal{C}} := \varphi(a_{\mathcal{C}})$  is a representative of the closed orbit  $X(\mathcal{C})$ . Now the facts that  $X_{\text{el}} \rightarrow X_{\text{an}}$  is locally trivial (Hilberts Satz 90) and that  $M(k)$  acts transitively on  $X_{\text{an}}(k)$  (Proposition 13.1) imply that  $M(k)\varphi(\overline{A}_k^+)$  is a neighborhood of every  $x_{\mathcal{C}}$  in  $\overline{X}_{\text{el}}(k)$ . Let  $\Omega_0 \subseteq U(k)$  be a compact neighborhood of 1 and  $\Omega_1 := \Omega_0 M(k)$ . Then the local structure theorem implies that  $\Omega_1$  is a compact subset of  $G(k)$  such that  $\Omega_1 \varphi(\overline{A}_k^+)$  is a neighborhood of each point  $x_{\mathcal{C}}$  in  $\overline{X}(k)$ .

Next we show that for every  $x \in \overline{X}(k)$  there is  $g \in G(k)$  such that  $gx$  lies in the open interior  $\Omega^0$  of  $\Omega_1 \varphi(\overline{A}_k^+)$ . This clearly implies the theorem because of compactness of  $\overline{X}(k)$  it can be covered by finitely many translates  $g\Omega^0$ .

Since  $U\overline{X}_{\text{el}}$  is (Zariski-)open in  $\overline{X}$  and meets every orbit there is  $g \in G(k)$  with  $gx \in U\overline{X}_{\text{el}}$ . But then there is  $u \in U(k)$  such that  $ugx \in \overline{X}_{\text{el}}$ . Then again since  $M(k)$  acts transitively on  $X_{\text{an}}(k)$  one can find  $m \in M(k)$  and  $a \in \overline{A}_k(k)$  with  $mugx = \varphi(a)$ . Finally, since every  $A_k$ -orbit of  $\overline{A}_k$  contains a fixed point  $a_{\mathcal{C}}$  in its closure there is a homomorphism  $\lambda : \mathbf{G}_m \rightarrow A_k$  such that  $\lim_{t \rightarrow 0} \lambda(t)a = a_{\mathcal{C}}$  for some  $\mathcal{C}$  (also in the Hausdorff topology). A positive power  $\lambda^n$  can be lifted to a homomorphism  $\tilde{\lambda} : \mathbf{G}_m \rightarrow A$ . Since then  $\lim_{t \rightarrow 0} \tilde{\lambda}(t)\varphi(a) = \varphi(\lim_{t \rightarrow 0} \lambda(t)^n a) = x_{\mathcal{C}}$  there is  $t \in \mathbf{G}_m(k)$  such that  $\tilde{\lambda}(t)mugx \in \Omega^0$ .  $\square$

**13.3. Remarks.** *i)* One can strengthen the weak polar decomposition in the following way: Let  $K \subseteq G(k)$  be a maximal compact subgroup in case  $k$  is archimedean and a compact open subgroup otherwise. Then  $\Omega$  can be chosen to be of the form  $\Omega = FK$  where  $F \subseteq G(k)$  is a finite subset. Indeed, if  $K$  is open then every compact subset of  $G(k)$  is a subset of finitely many translates of  $K$ . In the archimedean case see the argument in [KKSS15].



ii) For  $k = \mathbb{R}$  one has  $A_k^+ = A_2 \exp \mathfrak{a}_k^+$  where  $A_2 \subseteq A_k(k)$  is the subgroup of 2-torsion elements and  $\mathfrak{a}_k^+$  is the *compression cone*

$$(13.5) \quad \mathfrak{a}_k^+ = \{\xi \in \text{Lie } A_k(k) \mid d\sigma(\xi) \leq 0 \text{ for all } \sigma \in \Sigma_k\}.$$

We keep the notation of §12 on boundary degenerations. For local fields there is a simplification. More precisely, a given  $k$ -spherical variety  $X$  might in general have non-isomorphic boundary degenerations with the same spherical root system. This is due to the fact that the  $k$ -rational points of a  $G$ -orbit of  $\mathfrak{Y}$  may decompose into several  $G(k)$ -orbits. This does not happen for the small degeneration  $\tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{Y}}$  since  $\tilde{A} \cong \mathbf{G}_m^r$  and  $\tilde{\mathfrak{Y}} \cong \mathbf{A}^r$  with  $r = \text{rk}_k \mathfrak{Y}$ . Thus, the  $\tilde{A}(k)$ -orbits in  $\tilde{\mathfrak{Y}}(k)$  are parametrized by subsets of  $\Sigma_k$  with the fixed point corresponding to the empty set. By the weak polar decomposition every  $y \in \mathfrak{Y}(k)$  is  $G(k)$ -conjugate to an element of  $\tilde{\mathfrak{Y}}(k)$ . Thus we get:

**13.4. Corollary.** *Let  $k$  be a local field and  $X$  a  $k$ -spherical variety. Then every boundary degeneration  $\mathfrak{X}_y$  is isomorphic to a fiber of  $\tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{Y}}$ . In particular, two boundary degenerations  $\mathfrak{X}_{y_1}$  and  $\mathfrak{X}_{y_2}$  are isomorphic to each other if and only if  $Gy_1 = Gy_2 \subseteq \mathfrak{Y}$  if and only if  $\Sigma_k(\mathfrak{X}_{y_1}) = \Sigma_k(\mathfrak{X}_{y_2}) \subseteq \Sigma_k(X)$ .*

## 14. The Satake compactification

In this final section we compare our compactifications with compactifications already appearing in the literature.

Suppose the  $k$ -dense variety  $X$  is even  $K$ -convex. Then one can obtain another embedding by regarding  $X$  as a  $K$ -variety, take its standard embedding and observe that it is defined over  $k$ . The thus obtained variety, we denote it by  $X_{\text{st}}^K$ , is another canonical embedding of  $X$ . The advantages of  $X_{\text{st}}^K$  are that it is independent of  $k$  and that it is well studied. For symmetric varieties it has been introduced by DeConcini-Procesi in [DCP83] and more generally for  $K$ -spherical varieties by Luna-Vust [LV83]. If  $k = \mathbb{R}$  and  $X_{\text{st}}^{\mathbb{C}}$  is smooth then the set  $X_{\text{st}}^{\mathbb{C}}(\mathbb{R})$  of real points is clearly a compact  $G(\mathbb{R})$ -manifold containing  $X(\mathbb{R})$  as a dense open subset.

A major disadvantage of  $X_{\text{st}}^K$  is that it doesn't work well when  $X$  is not absolutely spherical because then  $X_{\text{st}}^K$  may contain infinitely many orbits and is therefore only defined up to certain birational isomorphisms. Also its boundary is not so easy to control. In particular, the union of all  $k$ -dense orbits may not be open (see Example 14.2 iii) further down).

The relation of the two standard embeddings is:

**14.1. Proposition.** *Let  $X$  be a  $k$ -spherical variety which we assume to be  $K$ -convex. Then its  $K$ -standard model  $X_{\text{st}}^K$  can be chosen to be defined over  $k$ . Moreover,  $X$  is also  $k$ -convex and the identity on  $X$  extends to a birational morphism*

$$(14.1) \quad \iota_X : X_{\text{st}}^k \rightarrow X_{\text{st}}^K.$$

The following are equivalent:

- i)  $\iota_X$  is injective on  $k$ -rational points.
- ii)  $\iota_X$  is an open embedding.
- iii)  $\mathcal{G}^*$  acts trivially on  $\Sigma_K(X) \setminus \Sigma_K^0(X)$  (see (10.6)).

When these conditions hold and  $X$  is  $k$ -spherical then  $\iota_X$  is bijective on  $k$ -rational points.

*Proof.* The  $k$ -standard embedding corresponds to the cone  $\mathcal{Z}_k$ . The same holds for  $K$ . Thus the existence of  $\iota_X$  follows from  $\mathcal{Z}_k \subseteq \mathcal{Z}_K$ . From toroidal theory follows that  $\iota_X$  is an open embedding if and only if  $\mathcal{Z}_k$  is a face of  $\mathcal{Z}_K$ . This holds if and only if the equations for  $\mathcal{N}_k$  do not contain an equation of the form  $\sigma = \gamma * \sigma$  (see Corollary 10.8). This shows the equivalence of *ii*) and *iii*). Now assume that  $\mathcal{Z}_k$  is not a face of  $\mathcal{Z}_K$ . Then there exists an ray  $\mathcal{R}$  of  $\mathcal{Z}_k$  which is not an extremal ray of  $\mathcal{Z}_K$ . Let  $\mathcal{C}$  be the smallest face of  $\mathcal{Z}_K$  containing  $\mathcal{R}$ . Then  $\iota_X$  maps the stratum  $X^k(\mathcal{R})$  of  $X_{\text{st}}^k$  onto the stratum  $X^K(\mathcal{C})$  of  $X_{\text{st}}^K$ . The fibers over  $k$ -rational points are tori whose dimension is  $\dim \mathcal{C} - \dim \mathcal{R} > 0$ . This shows that  $\iota_X$  is not injective on  $k$ -rational points. This shows the equivalence of *i*) and *ii*).

Finally, assume that  $\mathcal{Z}_k$  is a face of  $\mathcal{Z}_K$  and that  $X$  is  $k$ -spherical. Then the orbits of  $X_{\text{st}}^K$  containing a  $k$ -rational point are precisely those which correspond to a face of  $\mathcal{Z}_K$  which has an element of  $\mathcal{Z}_k$  in its relative interior. These are precisely the faces of  $\mathcal{Z}_k$ . Thus the image  $\iota_K$  contains all  $k$ -rational points of  $X_{\text{st}}^K$ .  $\square$

**14.2. Examples.** *i*) Condition *iii*) above is certainly satisfied if  $G$  is an inner form since then already the  $\mathcal{G}^*$ -action on  $S$  is trivial. It even suffices that  $\mathcal{G}^*$  acts trivially on  $S \setminus S^0$  which, in the real case, means that the Satake diagram of  $G$  does not contain any arrows (like all compact groups or  $SO(p, q)$  with  $p - q > 2$  and  $p - q \equiv 2 \pmod{4}$ ).

*ii*) Even if  $\mathcal{G}^*$  acts non-trivially on  $S \setminus S^0$  its action on  $\Sigma_K \setminus \Sigma_K^0$  may be trivial. That happens, e.g., for all Riemannian symmetric (see below) or the spaces  $U(p_1 + p_2, q_1 + q_2)/U(p_1, q_1) \times U(p_2, q_2)$  with  $k = \mathbb{R}$ .

*iii*) The simplest example where the two standard embeddings differ is when  $k = \mathbb{R}$  and  $X := SU(2, 1)/\mu_3 SO(2, 1)$  where  $\mu_3$  is the center of  $G$  (it is thrown in to make  $X_{\text{st}}^{\mathbb{C}}$  smooth). Then  $X_{\mathbb{C}}$  is the space of smooth quadrics in  $\mathbf{P}^3(\mathbb{C})$  and  $X_{\text{st}}^{\mathbb{C}}$  is the so-called space of complete quadrics. It is smooth and consists of four orbits: the open orbit  $X$ , two orbits  $D_1, D_2$  of codimension 1, and a closed orbit  $Y$ . The group  $SU(2, 1)$  is quasi-split and  $\mathcal{G}^*$  swaps the two simple roots of  $S$ . Thus, the two divisors  $D_1$  and  $D_2$  are also swapped by conjugation which means, in particular, they do not contain any real points. For the real points of the entire space this means  $X_{\text{st}}^{\mathbb{C}}(\mathbb{R}) = X(\mathbb{R}) \cup Y(\mathbb{R})$ . So all  $G(\mathbb{R})$ -orbits are either open or closed and the closed one,  $Y(\mathbb{R})$ , which is of codimension 2. On the other side, the closed orbit of  $X_{\text{st}}(\mathbb{R})$  is of codimension 1. This shows that

$$(14.2) \quad \iota_X : X_{\text{st}}(\mathbb{R}) \rightarrow X_{\text{st}}^{\mathbb{C}}(\mathbb{R})$$

is surjective but not injective. In fact,  $X_{\text{st}}$  is the blow-up of  $X_{\text{st}}^{\mathbb{C}}$  in  $Y$  with the proper transforms of  $D_1$  and  $D_2$  removed. This example also shows that the union of the two  $k$ -dense orbits, namely  $X \cup Y$ , is not open but only constructible.

**14.3. Remark.** In [BJ06], Borel and Ji describe various constructions for compactifying symmetric and locally symmetric spaces. In particular, §II.10 deals with the compactification of semisimple (aka. non-Riemannian or pseudo-Riemannian) symmetric spaces, i.e. real points of  $X = G/H$  with  $G$  real semisimple and  $H$  the fixed point group of an involution, by  $X_{\text{st}}^{\mathbb{C}}(\mathbb{R})$ . Unfortunately, the exposition is not quite accurate. In particular Thm. II.10.5 is wrong as stated. Part *iv*) of the theorem asserts for example that the boundary is of codimension 1 which is not true in general as Example 14.2 *iii*) above shows. The error seems to occur in the transition of (II.10.1) to (II.10.2) where unjustifiedly “ $\sigma$ ” is

replaced by “ $o$ ”. Coincidentally, the discussion from eq. (II.10.2) to the end of the section furnishes a description of our  $X_{\text{st}}(\mathbb{R})$  instead of  $X_{\text{st}}^{\mathbb{C}}(\mathbb{R})$ . In particular, [BJ06, Thm. II.10.5] becomes correct when interpreted as a description of the  $G(\mathbb{R})$ -orbits in  $X_{\text{st}}(\mathbb{R})$ .

For future reference we briefly make the connection of our theory to the classical theory of symmetric spaces of non-compact type. For this let  $k = \mathbb{R}$ , let  $G$  be a semisimple group, let  $\mathbf{G} := G(\mathbb{R})^0$  be the connected components of the real points, let  $\mathbf{K} \subseteq \mathbf{G}$  be a maximal compact subgroup, and let  $K \subseteq G$  be its Zariski closure in  $G$ . Then  $\mathbf{X} = \mathbf{G}/\mathbf{K}$  is a symmetric space and  $X := G/K$  is an  $\mathbb{R}$ -variety with  $\mathbf{X} \subseteq X(\mathbb{R})$ .

**14.4. Proposition.** *Let  $\mathbf{X} = \mathbf{G}/\mathbf{K}$  be a symmetric space. Assume that  $G$  is of adjoint type. Then the closure of  $\mathbf{X}$  in  $X_{\text{st}}(\mathbb{R})$  is the maximal Satake compactification of  $\mathbf{X}$ .*

*Proof.* The spherical roots of  $X$  are simply the restricted roots of  $G$  ([Vus74]). In particular, the  $\mathcal{G}^*$ -action on them is trivial. It follows that  $\mathbf{X}$  has isomorphic closures in  $X_{\text{st}}(\mathbb{R})$  and  $X_{\text{st}}^{\mathbb{C}}(\mathbb{R})$ . The closure in the latter is the maximal Satake compactification by [BJ06, I.17.9, II.14.10].  $\square$

It is well-known that the maximal Satake compactification of a symmetric space has the structure of manifold with corners. Recall, that a *manifold with corners* is a space  $\mathbf{X}$  with  $\mathcal{C}^\infty$ -structure which is locally modeled after  $\mathbb{R}_{\geq 0}^k \times \mathbb{R}^{n-k}$  with fixed  $n$  and varying  $k$ . See, e.g., [Joy12], for a nice introduction to manifolds with corners. The points corresponding to  $0^k \times \mathbb{R}^{n-k} \subseteq \mathbb{R}_{\geq 0}^k \times \mathbb{R}^{n-k}$  are said to have *depth*  $k$ . Let  $\mathbf{X}_k := \{x \in \mathbf{X} \mid \text{dp } x = k\}$ . A *boundary component* is the closure of a connected component of  $\mathbf{X}_1$ .

In this generality, pathologies may occur like selfintersections of boundary components with the teardrop ([Joy12, Fig. 2.1]) being the simplest example. For that reason, Jänich [Jän68] introduced the concept of an  $\langle N \rangle$ -manifold which is a manifold with corners together with an assignment of a number in  $\{1, \dots, N\}$  to every boundary component such that in every point all boundary components meeting there have a different number (see [Jän68, Joy12] for a precise definition). Clearly, the set  $\{1, \dots, N\}$  can be replaced by any set  $\Sigma$ . In that case, we talk about a  $\Sigma$ -manifold.

Now let  $\mathbf{X}$  be a  $\Sigma$ -manifold and  $\mathbf{Y}$  a boundary components of type  $\sigma$ . Then  $\mathbf{Y}$  is a  $\Sigma \setminus \{\sigma\}$ -manifold. More generally, let  $\mathbf{Y}_0$  be a connected component of  $\mathbf{X}_k$  for some  $k$ . Then  $\mathbf{Y}_0$  is contained in precisely  $k$  boundary components and let the elements of  $\Sigma_0 \subseteq \Sigma$  be their types. Then the closure of  $\mathbf{Y}$  of  $\mathbf{Y}_0$  has the structure of a  $\Sigma \setminus \Sigma_0$ -manifold. All subsets of the form  $\mathbf{Y}_0$  form a stratification which we call the *boundary stratification* of  $\mathbf{X}$ .

One can construct  $\Sigma$ -manifolds by cutting manifolds along hypersurfaces. More precisely, let  $\mathbf{X}'$  be a manifold (without boundary) and for each  $\sigma \in \Sigma$  let  $D_\sigma \subset \mathbf{X}'$  be a smooth hypersurface (possibly disconnected or empty). Assume that the  $D_\sigma$  intersect each other transversally. For each  $x \in \mathbf{X}'$  let  $\text{dp}(x)$  be the number of  $D_\sigma$ 's containing  $x$  and let  $\mathbf{X}'_0$  be set of points of depth 0, i.e., the complement of all  $D_\sigma$ . Then let  $\mathbf{X}$  be the manifold with corners obtained by cutting  $\mathbf{X}'$  along the  $D_\sigma$ . More precisely, a point of  $\mathbf{X}$  is a pair  $(x, \eta)$  with  $x \in \mathbf{X}'$  and  $\eta$  is a connected component of  $\mathbf{X}'_0$  near  $x$ , i.e., an element of the inverse limit  $\varprojlim \pi_0(U \cap \mathbf{X}'_0)$  where  $\pi_0$  denotes the set of connected components and  $U$  runs through all open neighborhoods of  $x$  in  $\mathbf{X}'$ . This limit has  $2^{\text{dp } x}$  elements, one for each orthant of a local chart in  $x$ . Thus, the map

$$(14.3) \quad \pi : \mathbf{X} \rightarrow \mathbf{X}_0 : (x, \eta) \mapsto x$$

is proper, surjective, and the fiber over  $x$  has cardinality  $2^{\text{dp } x}$ . Furthermore, each boundary component of  $\mathbf{X}$  maps to a specific hypersurface  $D_\sigma$  which makes  $\mathbf{X}$  into a  $\Sigma$ -manifold.

Now we apply this construction to a homogeneous  $\mathbb{R}$ -spherical variety  $X$  or, more precisely, to its real points  $X(\mathbb{R})$ . Choose any smooth fan  $\mathcal{F}$  whose support is  $\mathcal{Z}_{\mathbb{R}}(X)$  and let  $\mathbf{X}'$  be the set of real points of  $X(\mathcal{F})$ . Then  $\mathbf{X}'$  is a smooth compactification of  $X(\mathbb{R})$ . Its boundary is a union of smooth divisors indexed by the set  $\Sigma$  of 1-dimensional cones in  $\mathcal{F}$ . The manifold  $\mathbf{X}$  with corners obtained by cutting along the boundary is therefore a compact  $\Sigma$ -manifold. It is equivariant with respect to  $G(\mathbb{R})$ . One distinguishing feature of  $\mathbf{X}$  is that each connected component contains exactly one  $G(\mathbb{R})^0$ -orbit. If that orbit is a (Riemannian) symmetric space then the component is precisely its maximal Satake compactification. Observe that if  $X$  is  $k$ -wonderful one can choose  $\mathcal{F}$  to be the standard fan. In that case  $\Sigma$  can be identified with  $\Sigma_{\mathbb{R}}(X)$ .

**14.5. Proposition.** *Let  $\mathbf{G} := G(\mathbb{R})^0$ . Then the boundary strata of  $\mathbf{X}$  are precisely the  $\mathbf{G}$ -orbits.*

*Proof.* The boundary strata are connected and  $\mathbf{G}$ -stable. Moreover, the action of  $\mathbf{G}$  on each of them is locally transitive. Hence, they are  $\mathbf{G}$ -orbits.  $\square$

**14.6. Examples.** In all examples we have  $k = \mathbb{R}$  and  $G = SL(2, \mathbb{R})$ . All examples have  $\mathbb{R}$ -rank 1 and are not horospherical except in the last one. Hence  $X = G/H$  is compactified by adding a closed orbit  $Y$ .

*i)*  $H = SO(2)$ . Then  $G/H$  is the space of quadratic forms  $x_1\xi^2 + 2x_2\xi\eta + x_3\eta^2$  with discriminant  $x_2^2 - x_1x_3 = -1$ . Therefore  $X(\mathbb{R})$  is a 2-sheeted hyperboloid. Its closure in  $\mathbf{P}^3$  is the smooth quadric  $x_2^2 - x_1x_3 = -x_0^2$  with signature  $(3, 1)$ . Thus  $X_{\text{st}}(\mathbb{R}) \cong S^2$ , the 2-sphere. The boundary  $Y(\mathbb{R})$  is given by  $x_0 = 0$  which defines a great circle  $S^1 \subseteq S^2$  which separates  $X_{\text{st}}(\mathbb{R})$  in two parts  $X_{\pm}$ . Of course,  $X_{\text{st}}(\mathbb{R})$  can be identified with the Riemann sphere with  $X_{\pm}$  being the upper and lower halfplane. When we cut  $X_{\text{st}}(\mathbb{R})$  in  $Y(\mathbb{R})$  one gets a disjoint union of two closed disks. Thus,  $G(\mathbb{R})$  has two open and two closed orbits in  $\mathbf{X}$ .

*ii)*  $H = SO(1, 1)$ . In this case,  $G/H$  is the space of quadratic forms  $x_1\xi^2 + 2x_2\xi\eta + x_3\eta^2$  with discriminant  $x_2^2 - x_1x_3 = +1$ . Therefore  $X(\mathbb{R})$  is a 1-sheeted hyperboloid. Its closure in  $\mathbf{P}^3$  is the smooth quadric  $x_2^2 - x_1x_3 = x_0^2$  with signature  $(2, 2)$ . Thus  $X_{\text{st}}(\mathbb{R}) \cong \mathbf{P}^1(\mathbb{R}) \times \mathbf{P}^1(\mathbb{R}) \cong S^1 \times S^1$ , a 2-torus. The boundary  $Y(\mathbb{R})$  corresponds to the diagonal in  $S^1 \times S^1$  which does not separate  $X_{\text{st}}(\mathbb{R})$ . Cutting along  $Y(\mathbb{R})$  results in a cylinder  $S^1 \times [0, 1]$ . Thus  $G(\mathbb{R})$  has one open and two closed orbits in  $\mathbf{X}$ . Observe that  $\mathbf{X}$  is *not* the closure of an open  $G(\mathbb{R})$ -orbit in  $X_{\text{st}}(\mathbb{R})$ . This example also shows that connected components of  $\mathbf{X}$  can have more than one closed  $G(\mathbb{R})^0$ -orbit.

*iii)*  $H_1 = N_G(SO(2))$  or  $H_2 = N_G(SO(1, 1))$ . These two subgroups lead to the same  $\mathbb{R}$ -variety. In fact,  $X$  is the space similitude classes of quadratic forms  $x_1\xi^2 + 2x_2\xi\eta + x_3\eta^2$  with discriminant  $x_2^2 - x_1x_3 \neq 0$ . This shows that  $X_{\text{st}}(\mathbb{R})$  is the projective space  $\mathbf{P}^2(\mathbb{R})$ . The boundary is a smooth conic which separates  $\mathbf{P}^2(\mathbb{R})$  in two parts: one,  $\cong G/H_1$ , is diffeomorphic to an open disk, the other,  $\cong G/H_2$ , is diffeomorphic to an open Möbius strip. Cutting along  $Y(\mathbb{R})$  results in disjoint union of a closed disk (compactification of  $G/H_1$ ) and a closed Möbius strip (compactification of  $G/H_2$ ). Thus,  $G(\mathbb{R})$  has two open and two closed orbits in  $\mathbf{X}$ .

*iv)* In the last example let  $H = N$  be a maximal unipotent subgroup. It is horospherical, hence the compactification contains two closed orbits  $Y_1$  and  $Y_2$ . More precisely, if we take

for  $\mathcal{F}$  the two halflines in  $\mathcal{Z}_{\mathbb{R}}(X) = \mathbb{R}$  (there is no other possibility) then  $X(\mathcal{F})$  equals the projective space  $\mathbf{P}^2$  blown up at the origin. Thus,  $\mathbf{X}'$  is the non-orientable  $S^1$ -bundle over  $S^1$ , i.e., a Klein bottle. The two closed orbits form two sections of this bundle. Since the normal bundle is non-trivial, cutting along each of them does not result in two circles. Instead one gets a single circle which projects as a two-fold cover to  $S^1$ . Thus  $\mathbf{X}$  consists of one open orbit and two closed ones. Topologically,  $\mathbf{X}$  is a closed annulus. Note that the two closed orbits are no longer real points of algebraic varieties. Indeed, the isotropy group is the connected component  $P(\mathbb{R})^0$  which is a non-algebraic subgroup of  $SL(2, \mathbb{R})$ .

## References

- [Ber57] Marcel Berger, *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.
- [Bor91] Armand Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.
- [BJ06] Armand Borel and Lizhen Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2006.
- [BS64] Armand Borel and Jean-Pierre Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. **39** (1964), 111–164.
- [BT65] Armand Borel and Jacques Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.
- [Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.
- [BP15] Paolo Bravi and Guido Pezzini, *The spherical systems of the wonderful reductive subgroups*, J. Lie Theory **25** (2015), 105–123, [arxiv:1109.6777](https://arxiv.org/abs/1109.6777).
- [Bri87] Michel Brion, *Classification des espaces homogènes sphériques*, Compositio Math. **63** (1987), 189–208.
- [Bri90] Michel Brion, *Vers une généralisation des espaces symétriques*, J. Algebra **134** (1990), 115–143.
- [BLV86] Michel Brion, Domingo Luna, and Thierry Vust, *Espaces homogènes sphériques*, Invent. Math. **84** (1986), 617–632.
- [DCP83] Corrado De Concini and Claudio Procesi, *Complete symmetric varieties*, Invariant theory (Montecatini, 1982), Lecture Notes in Math., vol. 996, Springer, Berlin, 1983, pp. 1–44.
- [Die44] Jean Dieudonné, *Sur les fonctions continues  $p$ -adiques*, Bull. Sci. Math. (2) **68** (1944), 79–95.
- [Jän68] Klaus Jänich, *On the classification of  $O(n)$ -manifolds*, Math. Ann. **176** (1968), 53–76.
- [Joy12] Dominic Joyce, *On manifolds with corners*, Advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 21, Int. Press, Somerville, MA, 2012, pp. 225–258, [arxiv:0910.3518v2](https://arxiv.org/abs/0910.3518v2).
- [Kap50] Irving Kaplansky, *The Weierstrass theorem in fields with valuations*, Proc. Amer. Math. Soc. **1** (1950), 356–357.
- [Kem78] George Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), 299–316.
- [KKMSD73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973.
- [Kno93] Friedrich Knop, *Über Bewertungen, welche unter einer reduktiven Gruppe invariant sind*, Math. Ann. **295** (1993), 333–363.
- [Kno94] Friedrich Knop, *The asymptotic behavior of invariant collective motion*, Invent. Math. **116** (1994), 309–328.
- [Kno96] Friedrich Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. **9** (1996), 153–174.
- [KKS15a] Friedrich Knop, Bernhard Krötz, and Henrik Schlichtkrull, *The local structure theorem for real spherical varieties*, Compos. Math. **151** (2015), 2145–2159, [arxiv:1310.6390](https://arxiv.org/abs/1310.6390).
- [KKS15b] Friedrich Knop, Bernhard Krötz, and Henrik Schlichtkrull, *The tempered spectrum of a real spherical space* (2015), 52 p., [arxiv:1509.03429](https://arxiv.org/abs/1509.03429).

- [KKSS14] Friedrich Knop, Bernhard Krötz, Eitan Sayag, and Henrik Schlichtkrull, *Volume growth, temperedness and integrability of matrix coefficients on a real spherical space* (2014), 20 p., [arxiv:1407.8006](#).
- [KKSS15] Friedrich Knop, Bernhard Krötz, Eitan Sayag, and Henrik Schlichtkrull, *Simple compactifications and polar decomposition of homogeneous real spherical spaces*, *Selecta Math. (N.S.)* **21** (2015), 1071–1097, [arxiv:1402.3467](#).
- [Krä79] Manfred Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, *Compositio Math.* **38** (1979), 129–153.
- [Los09] Ivan Losev, *Demazure embeddings are smooth*, *Int. Math. Res. Not. IMRN* **14** (2009), 2588–2596.
- [Lun73] Domingo Luna, *Slices étales*, *Sur les groupes algébriques*, Soc. Math. France, Paris, 1973, pp. 81–105. *Bull. Soc. Math. France, Paris, Mémoire* **33**.
- [LV83] Domingo Luna and Thierry Vust, *Plongements d’espaces homogènes*, *Comment. Math. Helv.* **58** (1983), 186–245.
- [Mik86] Igor V. Mikityuk, *Integrability of invariant Hamiltonian systems with homogeneous configuration spaces*, *Mat. Sb. (N.S.)* **129(171)** (1986), 514–534, 591.
- [Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, vol. 15, Springer-Verlag, Berlin, 1988.
- [Ros56] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, *Amer. J. Math.* **78** (1956), 401–443.
- [Ros61] Maxwell Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, *Trans. Amer. Math. Soc.* **101** (1961), 211–223.
- [SV12] Yiannis Sakellaridis and Akshay Venkatesh, *Periods and harmonic analysis on spherical varieties* (2012), 291p., [arxiv:1203.0039](#).
- [Sat60] Ichirô Satake, *On representations and compactifications of symmetric Riemannian spaces*, *Ann. of Math. (2)* **71** (1960), 77–110.
- [Sat63] Ichirô Satake, *On the theory of reductive algebraic groups over a perfect field*, *J. Math. Soc. Japan* **15** (1963), 210–235.
- [Spr98] Tonny A. Springer, *Linear algebraic groups*, 2nd ed., *Progress in Mathematics*, vol. 9, Birkhäuser Boston, Inc., Boston, MA, 1998.
- [Sum75] Hideyasu Sumihiro, *Equivariant completion. II*, *J. Math. Kyoto Univ.* **15** (1975), 573–605.
- [Tit59] Jacques Tits, *Sur la classification des groupes algébriques semi-simples*, *C. R. Acad. Sci. Paris* **249** (1959), 1438–1440.
- [Tit66] Jacques Tits, *Classification of algebraic semisimple groups*, *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, Amer. Math. Soc., Providence, R.I., 1966, pp. 33–62.
- [VS13] Bart Van Steirteghem, *Various interpretations of the root system(s) of a spherical variety*, *Mini-workshop: Spherical Varieties and Automorphic Representations*, May 12–18, 2013, Oberwolfach Rep., Vol. 10, 2013, pp. 1464–1467.
- [VK78] Èrnest B. Vinberg and Boris N. Kimel’fel’d, *Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups*, *Funktsional. Anal. i Prilozhen.* **12** (1978), 12–19, 96.
- [Vus74] Thierry Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes*, *Bull. Soc. Math. France* **102** (1974), 317–333 (French).
- [Was96] Benjamin Wasserman, *Wonderful varieties of rank two*, *Transform. Groups* **1** (1996), no. 4, 375–403.