ON GENERICALLY SPLIT GENERIC FLAG VARIETIES

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Abstract. Let G be a split semisimple algebraic group over an arbitrary field F, let E be a G-torsor over F, and let P be a parabolic subgroup of G. The quotient variety $X := E/P$, known as a *flag variety*, is *generically split*, if the parabolic subgroup P is special. It is *generic*, provided that the G-torsor E over F is a standard generic G_k-torsor for a subfield $k \subset F$ and a split semisimple algebraic group G_k over k with $(G_k)_F = G$.

For any generically split generic flag variety X, we show that the Chow ring $\operatorname{CH} X$ is generated by Chern classes (of vector bundles over X). This implies that the topological filtration on the Grothendieck ring of X coincides with the computable gamma filtration. The results were already known in some cases including the case where P is a Borel subgroup.

We also provide a complete classification of generically split generic flag varieties and, equivalently, of special parabolic subgroups for split simple groups.

Contents

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0. Introduction

Let G be a split semisimple algebraic group over an arbitrary field k and let E be a standard generic G-torsor. By definition, E is the generic fiber of a standard versal G-torsor $U \to U/G$, whose total space U is a nonempty open G-equivariant subvariety in a finite-dimensional linear G-representation. The base of E is therefore the function field $F := k(U/G)$.

Let B be a Borel subgroup of G. There is a conjecture (see [12, Conjecture 1.1]) that for the variety $X := E/B$ – a generic variety of complete flags – the canonical epimorphism of the Chow ring $\mathrm{CH}\, X$ onto the associated graded ring $GK(X)$ of the topological filtration on the Grothendieck ring $K(X)$ is an isomorphism. Since the ring $K(X)$ is computed ([18]) and the topological filtration on it is known to coincide with the computable gamma filtration ([11, Example 2.4]), this conjecture, proved in several cases already, provides a way to compute the Chow ring.

There is an equivalent version of the above conjecture, where the Borel subgroup B is replaced by any fixed *special* parabolic subgroup P, see [13, Lemma 4.2]. By definition, P is special if any P-torsor over any extension field of the base field is trivial. This is equivalent to the fact that the generic flag variety $X := E/P$ is *generically split*, see Lemma 7.1.

In the present paper we show that a generically split generic flag variety $X = E/P$ has the same property as in the above particular case with $P = B$: the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration. This confirms that the introduced by A. Grothendieck gamma filtration is a good approximation of the topological filtration. And not only in the well-known sense that they both coincide after tensoring by $\mathbb{Q}$.

The reason for this coincidence (with $\mathbb{Z}$-coefficients) is our main result (Theorem 7.3) saying that the ring $\mathrm{CH}\, X$ coincides with its Chern subring. In its turn, Theorem 7.3 is a consequence of the similar statement for the Chow ring of the classifying space of P, see Proposition 5.5.

The assumption that P is special is essential. For instance, taking for G a spinor group $\mathrm{Spin}(n)$, we may take a parabolic subgroup $P \subset G$ such that the variety $X = E/P$ is a projective quadric. Then the gamma filtration on $K(X)$ is different from the topological filtration provided that $n \geq 9$. Indeed, as shown in [9], the component of codimension 2 of the graded ring associated with the gamma filtration contains an element of order 2; on the other hand, by [8, Theorem 6.1 and §3.1], the component of codimension 2 of the graded ring associated with the topological filtration is torsion-free for such n.

For an overview of sections of the paper we refer to the table of contents: the titles of sections are self-explaining. Note that the paper ends with a classification of special parabolic subgroups (or, equivalently, of generically split generic flag varieties).

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1. Special groups over algebraically closed fields

An affine algebraic group G over a field F is *special*, if any G-torsor over any extension field of F is trivial. Since for any field extension L/F, the set of isomorphism classes of G-torsors over L is identified with the Galois cohomology set $H^1(L,G)$, G is special if and only if $H^1(L,G)=1$ for any L.

Below in this section, we are assuming that the field F is algebraically closed.

Theorem 1.1 ([1, Exposé 5, Théoréme 3])**.** *Any special affine algebraic group over an algebraically closed field F is connected. A* semisimple *group G over F is special if and only if it is isomorphic to a finite direct product, where each factor is the special linear group* $\mathrm{SL}(n)$ *(for some $n\geq 2$) or the symplectic group* $\mathrm{Sp}(2n)$ *(for some $n\geq 2$).*

Classification of arbitrary special groups over F easily reduces to the semisimple case:

Lemma 1.2 (see also [7, Theorem 1])**.** *Let G be an affine connected algebraic group over an algebraically closed field F. The following three conditions are equivalent:*

1. *G is special;*
2. *the reductive group G/R_u is special, where R_u is the unipotent radical of G;*
3. *the semisimple group $(G/R_u)'$ is special, where $(G/R_u)'$ is the commutator subgroup of G/R_u.*

Proof. Let us prove equivalence of (1) and (2). By [2, Theorem 10.6], a connected unipotent group U over F has a finite filtration by closed normal subgroups with all successive quotients isomorphic to $\mathbb{G}_\mathrm{a}$. Since for any field L/F, $H^i(L,\mathbb{G}_\mathrm{a})=0$ for $i>0$, we get that $H^i(L,U)=0$ for $i>0$ provided that U is abelian. The derived series of the unipotent radical R_u of G is a finite filtration of R_u by closed connected characteristic subgroups with abelian successive quotients. It follows that the map of the set $H^1(L,G)$ of isomorphism classes of G-torsor over L to the set $H^1(L,G/R_u)$, induced by the projection $G\to G/R_u$, is a bijection.

Now we prove the equivalence of (2) and (3). We may assume that G is reductive, i.e. $G=G/R_u$. The quotient $S:=G/G'$ is then a (split) torus (called the *coradical* of G). By [2, Proposition 11.20], the image in S of a maximal torus T of G is a maximal torus of S and therefore equals S. The intersection $T':=T\cap G'$ is a (split) maximal torus of G'. For any field extension L/F, the exact sequence of tori

$$1\to T'\to T\to S\to 1$$

produces an exact sequence

$$T(L)\to S(L)\to H^1(L,T').$$

Since $H^1(L,T')=1$, the map $T(L)\to S(L)$ is surjective. It follows that the map $G(L)\to S(L)$ is also surjective. From the exact sequence

$$G(L)\to S(L)\to H^1(L,G')\to H^1(L,G)\to H^1(L,S)$$

with $H^1(L,S)=1$ we conclude that the map $H^1(L,G')\to H^1(L,G)$ is a bijection. □

2. Split special reductive groups over arbitrary fields

Let G be a split reductive group over an arbitrary field F and let G' be the split semisimple group given by the commutator subgroup of G. If G is special, then G remains special over any extension field of F. In particular, it remains special over an algebraic closure of F. By the results of the preceding section and classification of split semisimple groups, it follows that G' (over F) is isomorphic to a direct product of special linear and symplectic groups. The converse also holds:

Theorem 2.1 (see also [7, Proposition 15])**.** *For a split reductive group G over an arbitrary field the following statements are equivalent:*

(1) *G is special;*
(2) *G' is special;*
(3) *G' is isomorphic to a direct product of special linear and symplectic groups.*

Proof. The equivalence of (1) and (2) can be shown precisely the same way as the equivalence of (2) and (3) in Lemma 1.2. It has been shown that (2) implies (3) in the discussion before Theorem 2.1. Since special linear groups, symplectic groups, and direct products of special groups are special, (3) implies (2).

Let us also note that for a split reductive G, the image of a split maximal torus in G under the projection of G onto the coradical G/G' of G is a split maximal torus. Hence the torus G/G' is split and, in particular, special. This way Theorem 2.1 becomes a particular case of [7, Proposition 15]. □

3. Special parabolic subgroups of split semisimple groups

Let P be a parabolic subgroup of a split semisimple group G over an arbitrary field F. By [2, Proposition 20.5], the unipotent radical R_u of P is defined over F. Moreover, P contains a Levi subgroup L (over F); L is a split reductive group, the composition $L \hookrightarrow P \to P/R_u$ is an isomorphism.

The same proof as in Lemma 1.2 (equivalence of (1) and (2)) shows that P is special if and only if L is special. We should mention here that an arbitrary connected unipotent group U over F is known to possess a filtration by closed normal subgroups with quotients isomorphic to $\mathbb{G}_{\mathrm{a}}$ only in the case of perfect F, [4, IV, §2, Corollaire 3.9]. This is fine for $U = R_u$, because P and therefore its unipotent radical R_u are defined over the prime subfield of F, which is perfect.

By Theorem 2.1 we know that L is special if and only if the split semisimple group L' is special which means that L' is isomorphic to a direct product of special linear and symplectic groups.

4. Chow rings of classifying spaces

Let G be a split connected reductive algebraic group over an arbitrary field F. Let T be a split maximal torus of G and let $W := N_G(T)/T$ be the Weyl group of G with respect to T. We consider the G-equivariant Chow ring of the point $\mathrm{CH}_G(\operatorname{Spec} F)$ also called the Chow ring of the classifying space of G and introduced in [25].

Assume that G is special. According to [5, Proposition 6], the homomorphism

$$\mathrm{CH}_G(\operatorname{Spec} F) \to \mathrm{CH}_T(\operatorname{Spec} F),$$

induced by the imbedding $T \hookrightarrow G$ (see [5, §3.2]), identifies the ring $\mathrm{CH}_G(\operatorname{Spec} F)$ with the subring $\mathrm{CH}_T(\operatorname{Spec} F)^W$ of W-invariant elements in $\mathrm{CH}_T(\operatorname{Spec} F)$.

Viewing a character of T as a T-equivariant vector bundle over $\operatorname{Spec} F$, we can take its first equivariant Chern class in $\mathrm{CH}^1_T(\operatorname{Spec} F)$, [5, §3.2]. This assignment gives rise to a ring homomorphism

$$S(\hat{T}) \to \mathrm{CH}_T(\operatorname{Spec} F),$$

where $S(\hat{T})$ is the symmetric ring of the character group $\hat{T}$ of T. Since T is split, this homomorphism is a ring isomorphism so that we get an identification

$$\mathrm{CH}_G(\operatorname{Spec} F) = S(\hat{T})^W.$$

Summarizing we get

Theorem 4.1 ([5, Proposition 6])**.** *Let G be a special reductive group over a field F containing a split maximal torus T. Then $\mathrm{CH}_G(\operatorname{Spec} F) = S(\hat{T})^W$.*

5. Chern subrings for classifying spaces

Let G be an affine algebraic group over an arbitrary field F. The Chern subring of $\mathrm{CH}_G(\operatorname{Spec} F)$ is the subring generated by the equivariant Chern classes of finite-dimensional linear G-representations over F. Note that the representation ring $R(G)$ can be viewed as the G-equivariant Grothendieck ring of $\operatorname{Spec} F$, [17, Example 2]. The total equivariant Chern class $c : R(G) \to \mathrm{CH}_G(\operatorname{Spec} F)$ is a homomorphism of the additive group $R(G)$ into the multiplicative group of the elements in $\mathrm{CH}_G(\operatorname{Spec} F)$ with the homogeneous component of codimension 0 equal to 1.

For any reductive group G containing a split maximal torus T, the restriction homomorphism $R(G) \to R(T) = \mathbb{Z}[\hat{T}]$ identifies $R(G)$ with the ring $\mathbb{Z}[\hat{T}]^W$, [22, Théorème 4]. Since the addition in $\hat{T}$ becomes multiplication in $\mathbb{Z}[\hat{T}]$, we use the traditional exponential notation: we write e^χ for a character $\chi \in \hat{T}$ considered as an element of $\mathbb{Z}[\hat{T}]$. The above total Chern class c is a restriction of the total Chern class

$$c : \mathbb{Z}[\hat{T}] = R(T) \to \mathrm{CH}_T(\operatorname{Spec} F) = S(\hat{T}),$$

which is the homomorphism of the additive group of $\mathbb{Z}[\hat{T}]$ to the multiplicative group of the elements in $S(\hat{T})$ with the constant term 1, associating to $e^\chi \in \mathbb{Z}[\hat{T}]$ the element $1 + \chi \in S(\hat{T})$. In particular, the first Chern class of e^χ is $\chi \in S(\hat{T})$. Note that when we identify $\hat{T}$ with a subset in $S(\hat{T})$, the operation on $\hat{T}$ is the addition of the ring $S(\hat{T})$.

Example 5.1. For any integer $n \geq 2$, let us consider the special linear group $G := \mathrm{SL}(n)$. The group of characters $\hat{T}$ of a split maximal torus $T \subset G$ can be identified with the quotient of a free abelian group of rank n on generators $x_1, \dots, x_n$ modulo the subgroup generated by $x_1 + \dots + x_n$. Moreover, the Weyl group W is the symmetric group on $x_1, \dots, x_n$. Identifying $\mathrm{CH}_G(\operatorname{Spec} F)$ with $S(\hat{T})^W$ and using [14, Lemma 8.1] to compute the W-invariants, we get that $\mathrm{CH}_G(\operatorname{Spec} F)$ is the polynomial ring (over $\mathbb{Z}$) in the elementary symmetric polynomials $\sigma_2, \dots, \sigma_n$ in $x_1, \dots, x_n$. Since σ_i as an element of

$\mathrm{CH}_G(\operatorname{Spec} F)$ is the ith Chern class of the tautological representation of G (whose class in $R(G) = \mathbb{Z}[\hat{T}]^W$ is equal to $e^{x_1} + \cdots + e^{x_n}$), we see that the ring $\mathrm{CH}_G(\operatorname{Spec} F)$ coincides with its Chern subring.

Example 5.2. For any integer $n \geq 1$, let us consider the symplectic group $G := \mathrm{Sp}(2n)$. The group of characters $\hat{T}$ of a split maximal torus $T \subset G$ can be identified with a free abelian group of rank n on generators $x_1, \ldots, x_n$. Moreover, the Weyl group W is a semidirect product of the symmetric group on $x_1, \ldots, x_n$ by a direct product of n copies of $\mathbb{Z}/2\mathbb{Z}$, where the ith copy of $\mathbb{Z}/2\mathbb{Z}$ changes the sign of x_i and acts trivially on the remaining generators. Identifying $\mathrm{CH}_G(\operatorname{Spec} F)$ with $S(\hat{T})^W$, we see that $\mathrm{CH}_G(\operatorname{Spec} F)$ is the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ in the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ in squares of $x_1, \ldots, x_n$. Since σ_i is the $2i$th Chern class of the tautological representation of G (whose class in $R(G) = \mathbb{Z}[\hat{T}]^W$ is equal to $e^{x_1} + e^{x_1^{-1}} + \cdots + e^{x_n} + e^{x_n^{-1}}$ so that the Chern classes are the $2n$ elementary symmetric polynomials in $\pm x_1, \ldots, \pm x_n$), we see that the ring $\mathrm{CH}_G(\operatorname{Spec} F)$ is generated by Chern classes. Note that the odd Chern classes of the tautological representation vanish.

Example 5.3. Let $G = G_1 \times \cdots \times G_n$ with $n \geq 1$ and with each G_i isomorphic to a symplectic or a special linear group. Let T_i be a split maximal torus in G_i. Then $T := T_1 \times \cdots \times T_n$ is a split maximal torus in G and the Weyl group W of G relative to T is the product $W_1 \times \cdots \times W_n$ with W_i being the Weyl group of G_i relative to T_i. It follows that

$$\mathrm{CH}_G(\operatorname{Spec} F) = S(\hat{T})^W = S(\hat{T}_1)^{W_1} \otimes \cdots \otimes S(\hat{T}_n)^{W_n} = \\ \mathrm{CH}_{G_1}(\operatorname{Spec} F) \otimes \cdots \times \mathrm{CH}_{G_n}(\operatorname{Spec} F).$$

Since for any i the ring $\mathrm{CH}_{G_i}(\operatorname{Spec} F)$ coincides with its Chern subring, the same holds for $\mathrm{CH}_G(\operatorname{Spec} F)$.

Let G be a split reductive group over F and let G' be its commutator subgroup. The following lemma shows that the homomorphism of the representation rings $R(G) \to R(G')$, given by restriction of representations, is surjective (cf. the beginning of proof of Proposition 5.5).

Lemma 5.4 (A. S. Merkurjev, c.f. [16, Lemma 5.2])**.** *Let M be an abelian group endowed with an action of a finite (abstract) group W and let $M'' \subset M^W$ be a torsion-free subgroup of W-invariant elements. For $M' := M/M''$, the homomorphism $\mathbb{Z}[M]^W \to \mathbb{Z}[M']^W$ is surjective.*

Proof. For any subgroup $W' \subset W$, since the action of W' on M'' is trivial, we have $H^1(W', M'') = \operatorname{Hom}(W', M'')$. Since W' is finite and the group M'' is torsion-free, we have $\operatorname{Hom}(W', M'') = 0$. The exact cohomology sequence

$$M^{W'} \longrightarrow (M')^{W'} \longrightarrow H^1(W', M') = 0$$

therefore tells us that the homomorphism $M^{W'} \to (M')^{W'}$ is surjective. For any $m' \in M'$, the sum $s_{m'} \in \mathbb{Z}[M']$ of the exponents of the elements of the W-orbit of m' in M' is W-equivariant. The additive group of $\mathbb{Z}[M']^W$ is generated by all such $s_{m'}$.

So, we only need to show that $s_{m'}$ is in the image of $\mathbb{Z}[M]^W$ for any given $m' \in M'$. For this, we consider the stabilizer $W' \subset W$ of m'. By surjectivity of $M^{W'} \to (M')^{W'}$, we can find a W'-invariant element $m \in M$ mapped to m'. The W-orbit of $e^m \in \mathbb{Z}[M]$ produces an element $s_m \in \mathbb{Z}[M]^W$ mapped to $s_{m'}$. □

Proposition 5.5. *For any split special reductive group G over an arbitrary field, the ring $\mathrm{CH}_G(\operatorname{Spec} F)$ coincides with its Chern subring.*

Proof. Let T be a split maximal torus in G and let T' be the split maximal torus $T \cap G'$ of G'. The Weyl group W of G relative to T is identified with the Weyl group of G' relative to T', and this identification respects the action on T'. Moreover, the action of W on T/T' is trivial. Therefore, setting $M := \hat{T}$ and $M'' := \widehat{T/T'}$, we reduce the proof to the following elementary statement (see Lemma 5.6 below).

Let M be a lattice endowed with an action of a finite group W. We define the (abstract) total Chern class map $c : \mathbb{Z}[M] \to S(M)$ as the homomorphism of the additive group $\mathbb{Z}[M]$ into the multiplicative group of the elements in $S(M)$ with the homogeneous component of degree 0 equal to 1, assigning to $e^m \in \mathbb{Z}[M]$ for each $m \in M$ the element $1+m \in S(M)$. Note that $c(\mathbb{Z}[M]^W) \subset S(M)^W$. We also write c for the map $\mathbb{Z}[M]^W \to S(M)^W$ thus obtained, and also call it the (abstract) total Chern class map. We say that the ring $S(M)^W$ is generated by Chern classes, if it is generated by the homogeneous component of the elements from the image of c.

Lemma 5.6. *Let M be a lattice endowed with an action of a finite group W and let $M'' \subset M^W$ be a subgroup of W-invariant elements such that the quotient $M' := M/M''$ is free. Assume that the ring $S(M')^W$ is generated by Chern classes. Then the ring $S(M)^W$ is also generated by Chern classes.*

Proof. Choosing a basis of M'', taking the corresponding flag of subgroups in M'', and decomposing the projection $M \to M'$ in a finite chain of epimorphisms with cyclic kernels, we reduce the proof to the case of cyclic M''.

In the commutative diagram

$$\begin{array}{ccc} R(G) = \mathbb{Z}[M]^W & \longrightarrow & R(G') = \mathbb{Z}[M']^W \\ \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle c} \\ \mathrm{CH}_G(\operatorname{Spec} F) = S(M)^W & \longrightarrow & \mathrm{CH}_{G'}(\operatorname{Spec} F) = S(M')^W \end{array}$$

the upper map is an epimorphism by Lemma 5.4. We fix a generator y of M''. We show that a homogeneous element $f \in S(M)^W$ is a polynomial in Chern classes using induction on $\deg f$. For f of degree 0 there is nothing to prove. Assume that the degree of f is positivie and write its image $f' \in S(M')^W$ as a polynomial in Chern classes of some elements in $\mathbb{Z}[M']^W$. Lifting these elements to $\mathbb{Z}[M]^W$ in an arbitrary way and taking the same polynomial in the Chern classes of the liftings, we get an element $g \in S(M)^W$ such that the difference $f - g$ vanishes in $S(M')^W$. Since the kernel of the ring homomorphism $S(M) \to S(M')$ is the ideal in $S(M)$ generated by y, we can write $f - g = yh$ with some homogeneous $h \in S(M)$ of degree $\deg h = \deg f - 1$. Since y and yh are W-invariant while the ring $S(W)$ is a polynomial ring, it follows that $h \in S(M)^W$. So, by induction hypothesis, h can be written as a polynomial in Chern classes. □

□

6. Chern subrings for special parabolic subgroups

Proposition 6.1. *Let P be a special parabolic subgroup of a split semisimple algebraic group over an arbitrary field F. Then the ring $\mathrm{CH}_P(\operatorname{Spec} F)$ coincides with its Chern subring.*

Proof. Let L be a Levi subgroup of P. Then L is a split special reductive group so that the ring $\mathrm{CH}_L(\operatorname{Spec} F)$ coincides with its Chern subring. Therefore it suffices to show that the homomorphism $\mathrm{CH}_P(\operatorname{Spec} F) \to \mathrm{CH}_L(\operatorname{Spec} F)$ is an isomorphism.

In order to show that $\mathrm{CH}^i_P(\operatorname{Spec} F) \to \mathrm{CH}^i_L(\operatorname{Spec} F)$ is an isomorphism for any $i \geq 0$, we recall that $\mathrm{CH}^i_P(\operatorname{Spec} F) = \mathrm{CH}^i(U/P)$ for an open subset $U \subset V$ of a finite-dimensional linear P-representation V admitting a P-torsor $U \to U/P$ and such that $\operatorname{codim}_V(V \setminus U) \geq i$. For any such U we also have $\mathrm{CH}^i_L(\operatorname{Spec} F) = \mathrm{CH}^i(U/L)$ and the homomorphism $\mathrm{CH}^i(U/P) \to \mathrm{CH}^i(U/L)$ we are talking about is the pull-back with respect to the projection $U/L \to U/P$.

Let R_u be the unipotent radical of P. Then P is the semidirect product $R_u \rtimes L$. In particular, the quotient variety P/L is isomorphic to the underlying variety of R_u. Since the algebraic group R_u is connected, unipotent, and defined over the prime subfield of F, the variety R_u is isomorphic to an affine space, [4, IV, §2, Corollaire 3.9 and IV, §4, Théorème (de Lazard) 4.1].

Since P is special, for any point $x \in U/P$ the fiber of the morphism $U \to U/P$ over x is isomorphic to $P_{F(x)}$. Therefore the fiber over x of the morphism $U/L \to U/P$ is isomorphic to the quotient variety $(P/L)_{F(x)}$. As explained above, P/L is an affine space so that the pull-back of Chow groups $\mathrm{CH}(U/P) \to \mathrm{CH}(U/L)$ is an isomorphism by the homotopy invariance of Chow groups, [6, Theorem 57.13]. □

7. Chern subring for generic generically split flag varieties

Let F be a field, G a split semisimple algebraic group over F, P a parabolic subgroup of G, and E a G-torsor over F. The flag variety $X := E/P$ is *generically split* (cf. [20] and [21]) if the G-torsor E becomes trivial over the function field $F(X)$.

The flag variety X is *generic*, provided that the G-torsor E over F is a standard generic G_k-torsor for a subfield $k \subset F$ and a split semisimple algebraic group G_k over k with $(G_k)_F = G$.

The following lemma can be used to detect the special parabolic subgroups (cf. §8):

Lemma 7.1 (cf. [13, Lemma 4.1])**.** *A generic flag variety $X = E/P$ is generically split if and only if the parabolic subgroup P is special.*

Proof. If P is special, the variety E'/P is generically split for any G-torsor E' over F by [14, Lemma 6.5]. In particular, this holds for $E' = E$.

Conversely, if a generic flag variety $X = E/P$ is generically split, then for any field L/F and any G-torsor E' over L, the flag variety E'/P is also generically split. It follows by [19, §5] that P is special.

Below is a direct proof of both implications due to A. Merkurjev.

Assume that P is special. The pull-back of a G-torsor E over the field F to the field $F(E/P)$ coincides with the push-forward of the generic fiber of the P-torsor $E \to E/P$ and hence is trivial since P is special.

Conversely, assume that a generic flag variety E/P is generically split. We may assume that G is defined over a subfield $k \subset F$ such that the G-torsor E over F is the generic fiber of the G-torsor $U \to U/G$ for a nonempty open subset U of a finite-dimensional linear G-representation over k. In particular, F is the function field $k(U/G)$. By assumption, the pull-back E' of E to the field $F(E/P)$ is trivial. On the other hand, E' is the push-forward of the generic fiber E'' of the P-torsor $U \to U/P$. Since for any field L/F the map $G(L) \to (G/P)(L)$ is surjective by [2, Proposition 20.5], the map $H^1(L,P) \to H^1(L,G)$ has trivial kernel. It follows that E'' is trivial, i.e., the morphism $U \to U/P$ has a section over a nonempty open subset $W \subset U/P$. Since the P-torsor $U \to U/P$ is versal, every P-torsor over an infinite field K/k is isomorphic to the fiber of $U \to U/P$ over a point in $W(K)$ ([23, §5 of Chapter I]) and hence is trivial. Therefore, P is special. □

Here is an extended version of Lemma 7.1:

Corollary 7.2. *Let G be a split semisimple algebraic group over an arbitrary field k, let P be a parabolic subgroup of G, and let $\tilde{E}$ be a standard generic G-torsor. The following statements are equivalent:*

(1) *The parabolic subgroup P is special.*
(2) *The generic flag variety $\tilde{E}/P$ is generically split.*
(3) *For any field F/k and any G-torsor E over F, the flag variety E/P is generically split.*
(4) *For any field F/k and any non-trivial G-torsor E over F, the flag variety E/P has no rational points.*

Proof. Equivalence of (1), (2) and (3) is already contained in the statement or in the proof of Lemma 7.1.

If E/P is generically split and has a rational point, then the G-torsor E becomes trivial over the function field $F(E/P)$ and this function field is purely transcendental. It follows that E is trivial already over F. Therefore (3) implies (4).

Conversely, let us assume (4). Since over $F(E/P)$ the variety E/P gets a rational point, the G-torsor E becomes trivial over $F(E/P)$. This means that the flag variety E/P is generically split. □

Here is the main result of the paper:

Theorem 7.3. *For any generically split generic flag variety X over a field F, the Chow ring $\mathrm{CH}\,X$ coincides with its Chern subring.*

Proof. The Chow ring $\mathrm{CH}\,X = \mathrm{CH}(E/P)$ can be viewed as the P-equivariant Chow ring $\mathrm{CH}_P(E)$, [5]. The characteristic map $\mathrm{CH}_P(\mathrm{Spec}\,F) \to \mathrm{CH}_P(E)$ is then the pull-back with respect to the structure morphism $E \to \mathrm{Spec}\,F$. It follows that the characteristic map maps the Chern subring of $\mathrm{CH}_P(\mathrm{Spec}\,F)$ into the Chern subring of $\mathrm{CH}\,X$. Since the characteristic map is surjective (see, e.g., [11, §2]) and the ring $\mathrm{CH}_P(\mathrm{Spec}\,F)$ coincides with its Chern subring (Proposition 6.1), the statement follows. □

Corollary 7.4. *For any generic generically split flag variety X, the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration.*

Proof. For any regular algebraic variety X over a field with the property that the ring $\mathrm{CH}\,X$ coincides with its Chern subring, the topological filtration on $K(X)$ coincides with the gamma filtration, cf. [10, Proof of Theorem 3.7]. More precisely, gamma filtration coincides with the topological one if and only if the graded ring $GK(X)$ associated with the topological filtration coincides with its Chern subring. □

8. Classification of special parabolic subgroups

In this section we classify special parabolic subgroups of split simple algebraic groups G. In view of Lemma 7.1, this is equivalent to a classification of generically split generic flag varieties. For G of classical types, we indicate an alternative description of these varieties.

As is already clear from the description in §3, the classification does not depend on the base field.

Since parabolic subgroups of products of semisimple groups are products of parabolic subgroups of the factors, which are special if and only if each factor is special, we also get a classification of special parabolic subgroups for split semisimple groups which are products of simple ones. This includes all adjoint and all simply connected semisimple groups.

Let G be a split semisimple algebraic group over an arbitrary field F. To a parabolic subgroup P of G, a subset of the set of vertices of the Dynkin diagram of G is associated, [24, §1.6]. For instance, the empty subset is associated to a Borel subgroup. The subset associated to P is called the *type of P*. Most of the time, the classification is easier to formulate in terms of the complement of P which we call the *co-type of P*.

Recall that the Dynkin diagram of the semisimple part of (a Levi subgroup of) P is obtained by removing from the Dynkin diagram of G the vertices of the co-type of P (as well as the edges attached to at least one vertex removed). By §3, a necessary condition for P to be special is that the Dynkin diagram of its semisimple part is a union of diagrams of types A and/or C. This necessary condition turns also out to be sufficient for simply connected G.

Corollary 7.2 relates detection of special parabolic subgroups in G with the possible Tits indexes of the inner twisted forms of G. Indeed, if E is a G-torsor over F, the flag variety E/P is a projective homogeneous variety under the twisted form G^E of G given by E. Note that the group G^E is split (i.e., $G^E \simeq G$) if and only if the torsor E is trivial: if G^E is split, the variety E/B for a Borel subgroup $B \subset G$ has a rational point therefore E is trivial since B is special.

Recall that the Tits index of G^E is, like the type of P, also a subset of vertices of the Dynkin diagram of G. In terms of the Tits indexes, the equivalence (1) ⇔ (4) in Corollary 7.2 translates as follows: P is special if and only if every Tits index that occurs for G^E when E varies over all G-torsors over all field extensions of F, coincides with the total Dynkin diagram provided that it contains the co-type of P.

From now on G is supposed to be simple split. The part of the classification of Tits indexes given in [24], which deals with the groups of inner type, provides the list of all

Tits indexes of G^E in the case of adjoint G. The classification given below in the adjoint case is therefore justified by this information. For non-adjoint G, however, the list of possible Tits indexes of G^E is a subset of the adjoint list, enlarging the list of co-types of special parabolic subgroups and requiring additional arguments.

In fact, for G of classical types, the most natural argument seems to rely on realizations of the group G and the varieties E/P. This argument, which we provide, covers the adjoint case as well.

For G of exceptional types, we do not comment on the adjoint case anymore and we apply results of [20] in the non-adjoint case (occurring for the types E_6 and E_7 only).

We use the numbering of the vertices of the Dynkin diagram of G as in [3]. (For classical types, this numbering is also described in [15, §24.A].) This way the co-type of P becomes a set of integers.

We proceed with the classification by the type of G:

8a. **Type** A. Let G be of type A_n ($n \geq 1$). Then G is isomorphic to $\mathrm{SL}(n+1)/\mu_m$, where $m \geq 1$ is a divisor of $n+1$. Let $r_1 < \cdots < r_k$ be the co-type of P. Then

- *P is special if and only if the greatest common divisor of the integers $r_1, \dots, r_k$ and m is* 1.

In particular, all parabolic subgroups (including $G = \mathrm{SL}(n+1)$ itself) are special if $m = 1$.

Note that a G-torsor E over F gives rise to a central simple F-algebra A of degree $n+1$ and of exponent dividing m. The isomorphism class of any central simple F-algebra A of degree $n+1$ and of exponent dividing m can be obtained this way. Moreover, the algebra given by E is split if and only if E is trivial. The variety E/P can be identified with the variety of flags of right ideals in A of reduced dimensions $r_1, \dots, r_k$.

To proof the above classification of special parabolic subgroups in G, we use Corollary 7.2. If the numbers $r_1, \dots, r_k, m$ have a common divisor $d > 1$, we find – over an appropriate field L/F – a central simple algebra A of degree $n+1$ and of index and exponent d. Let E be a G-torsor over L producing the algebra A. Since $\operatorname{ind} A = d \neq 1$, E is nontrivial. However the variety E/P of flags of right ideals in A of reduced dimensions $r_1, \dots, r_k$ has a rational point. Consequently, P is not special.

On the other hand, if the greatest common divisor of $r_1, \dots, r_k, m$ is 1, then for any field L/F and any G-torsor E over L, the index of the central simple L-algebra A associated to E is 1 provided that the flag variety E/P possesses a rational point. This means that E is trivial. Therefore P is special.

8b. **Type** B. Let G be adjoint of type B_n ($n \geq 1$). Then G is isomorphic to the special orthogonal group $O^+(2n+1)$ (see [15, §23] for the definition of $O^+(2n+1)$ in arbitrary characteristic).

- *A parabolic subgroup P of G is special if and only if its co-type contains the number n.*

Now assume that G is simply connected of type B_n. Then G is isomorphic to the spinor group $\mathrm{Spin}(2n+1)$.

- *A parabolic subgroup P of G is special if and only if at least one of the numbers $n, n-1, n-2$ is not included in the type P.*

(For $n \geq 3$, the condition means that at least one of $n, n-1, n-2$ belongs to the co-type.) In particular, any parabolic subgroup (including G) is special if $n \leq 2$; for $n = 3$ any parabolic subgroup different from G is special.

In both (adjoint and simply connected) cases, a G-torsor E over F gives rise to a non-degenerate quadratic form q of dimension $2n+1$ over F. (The even Clifford algebra of q being split in the simply connected case.) In the adjoint case, we have this way a bijection between the set of isomorphism classes of G-torsors over F and the set of similarity classes of non-degenerate $2n+1$-dimensional quadratic forms over F, where the trivial torsor corresponds to (the similarity class of) a quadratic form of the (maximal possible) Witt index n. In the simply connected case, we get a surjective map with trivial kernel, where the destination set is given by the quadratic forms whose even Clifford algebra is split.

In any case, if $r_1 < \cdots < r_k$ is the co-type of P, the variety E/P can be identified with the variety of flags of totally isotropic subspaces of q of dimensions $r_1, \ldots, r_k$.

Here is the proof of the above classification for the adjoint case. If n is not included in the co-type of P, we consider a quadratic form q of dimension $2n+1$ and of Witt index $n-1$ (existing over an appropriate field L/F). The corresponding G torsor E over L is then nontrivial, but the variety E/P has a rational point. So, P is not special in this case.

If n is included in the co-type of P and for some G-torsor E over some field L/F, the variety E/P has a rational point, then the Witt index of the corresponding quadratic form q is n meaning that E is trivial. It follows that P is special in this case.

And here is the proof of the above classification for the simply connected case. If all of $n, n-1, n-2$ are in the type of P, then $n \geq 3$ and we find a quadratic form q of dimension $2n+1$ and of Witt index $n-3$ (over an appropriate field L/F) such that the even Clifford algebra of q is split. The corresponding G torsor E over L is then nontrivial, but the variety E/P has a rational point. So, P is not special in this case.

If at least one of $n, n-1, n-2$ is not in the type of P and for some G-torsor E over some field L/F, the variety E/P has a rational point, then the Witt index of the corresponding quadratic form q is at least $n-2$. Since the even Clifford algebra of q is split, the Witt index of q is actually n so that E is trivial. It follows that P is special in this case.

8c. **Type** C**.** For simply connected G of type C_n ($G \simeq \mathrm{Sp}(2n)$), since G is special,

- *all parabolic subgroups (including G itself) are special.*

Let G be adjoint of type C_n ($n \geq 1$). Then G is isomorphic to the projective symplectic group $\mathrm{PGSp}(2n)$.

- *A parabolic subgroup P of G is special if and only if its co-type contains at least one odd number.*

A G-torsor E over F gives rise to a central simple F-algebra A of degree $2n$ endowed with a symplectic involution σ. Note that, since A admits a symplectic involution, the index of A is a 2-power.

This way we get a bijection between the set of isomorphism classes of G-torsors over F and the set of isomorphism classes of the pairs (A, σ), where the trivial torsor corresponds to the split algebra. If $r_1 < \cdots < r_k$ is the co-type of P, the variety E/P can be identified with the variety of flags of right σ-isotropic ideals in A of reduced dimensions $r_1, \ldots, r_k$.

Assume that the co-type of P consists of even numbers only. Consider an appropriate field L/F and an endowed with a symplectic involution σ central simple L-algebra A of degree $2n$ and index 2 such that the index of (A, σ) in the sense of [15, §6.A] is the set of all even integers from 0 to n. (For even n, the involution σ is hyperbolic; for odd n it is "almost hyperbolic".) Let E be the corresponding G-torsor over L. The variety E/P then has a rational point, but E is not trivial (because $\operatorname{ind} A \neq 1$). We see that P is not special in this case.

If in the co-type of P there is at least one odd number, any E (over any L/F) with the property $(E/P)(L) \neq \emptyset$ will be trivial, because the index of the corresponding central simple L-algebra will have to be odd (divisor of a 2-power) and therefore equal to 1.

8d. **Type** D. We start with $G = O^+(2n)$, $n \geq 3$.

- *A parabolic subgroup P of G is special if and only if its co-type contains at least one of the numbers n and $n-1$.*

Isomorphism classes of G-torsors over F are in natural bijection with the isomorphism classes of the $2n$-dimensional non-degenerate quadratic forms over F of trivial discriminant. Under this bijection, the trivial G-torsor corresponds to the hyperbolic quadratic form. If a G-torsor E corresponds to a quadratic form q and the co-type of P consists of the numbers $r_1 < \cdots < r_k$, the flag variety E/P is interpreted as follows. If $r_k \leq n-2$, E/P is the variety of flags of totally isotropic subspaces in q of dimensions $r_1, \ldots, r_k$. If $r_k = n$ and $r_{k-1} = n-1$, then E/P is the variety of flags of totally isotropic subspaces in q of dimensions $r_1, \ldots, r_{k-1}$. Finally, if $r_{k-1} \leq n-2$ and $r_k \geq n-1$, E/P is one of two (isomorphic) connected components of the variety of flags of totally isotropic subspaces in q of dimensions $r_1, \ldots, r_{k-1}, n$.

Now we prove the above classification. If none of n and $n-1$ is in the co-type of P (meaning that $r_k \leq n-2$), we show that P is not special, considering a field L/F admitting a $2n$-dimensional quadratic form q of trivial discriminant and of Witt index $n-2$. For the corresponding E, the variety E/P has then a rational point, but E is not trivial.

If at least one of $n, n-1$ is in the co-type of P and E/P has a rational point for some E over some L/F, the Witt index of the quadratic form q corresponding to E is at least $n-1$. Since the discriminant of q is trivial, it follows that the Witt index is n, i.e., q is hyperbolic. Therefore E is trivial and we proved P is special.

Next we consider $G = \operatorname{Spin}(2n)$ (the simply connected group of type D_n).

- *A parabolic subgroup P of G is special if and only if at least one of the numbers n, $n-1$, $n-2$, $n-3$ is not in the type of P.*

(For $n \geq 4$, the condition means that an least one of $n, n-1, n-2, n-3$ is in the co-type of P.) In particular, for $n = 3$ all parabolic subgroups are special (including $G = \operatorname{Spin}(6) = \operatorname{SL}(4)$); for $n = 4$, all parabolic subgroups different from G are special.

Isomorphism classes of G-torsors over F map surjectively and with trivial kernel onto the isomorphism classes of the $2n$-dimensional non-degenerate quadratic forms over F with trivial discriminant and split Clifford algebra. The interpretation of the flag varieties is the same as in the case of $G = O^+(2n)$.

To prove the classification, we first assume that all of n, $n-1$, $n-2$, $n-3$ are in the type of P. We show that P is not special, considering a field L/F admitting a $2n$-dimensional quadratic form q with trivial discriminant, split Clifford algebra, and Witt index $n-4$. For the corresponding E, the variety E/P has then a rational point, but E is not trivial.

If at least one of $n, n-1, n-2, n-3$ is not in the type of P and E/P has a rational point for some E over some L/F, the Witt index of the quadratic form q corresponding to E is at least $n-3$. Since the discriminant of q is trivial and the Clifford algebra of q is split, it follows that q is hyperbolic. Therefore E is trivial and P is special.

Our next case is the adjoint one: $G = \mathrm{PGO}^+(2n)$.

- *A parabolic subgroup P of G is special if and only if its co-type satisfies at least one of the following three conditions:*
  - *(1) contains an odd number $\leq n-2$ and at least one of the numbers $n-1, n$;*
  - *(2) contains both of the numbers $n-1, n$;*
  - *(3) for odd n, contains at least one of the numbers $n-1, n$.*

We have a canonical surjection with trivial kernel of the set of isomorphism classes of G-torsors over F onto the set of isomorphism classes of central simple F-algebras of degree $2n$ endowed with a quadratic pair with trivial discriminant, [15, §29.F]. If a G-torsor E corresponds to an algebra A and the co-type of P consists of the numbers $r_1 < \cdots < r_k$, the flag variety E/P is interpreted as a flag variety of isotropic right ideals in A of the reduced dimensions described as follows. If $r_k \leq n-2$, the reduced dimensions are $r_1, \ldots, r_k$. If $r_k = n$ and $r_{k-1} = n-1$, the reduced dimensions are $r_1, \ldots, r_{k-1}$. Finally, if $r_{k-1} \leq n-2$ and $r_k \geq n-1$, the variety E/P is one of two (not necessarily isomorphic) connected components of the variety of flags of isotropic ideals in A of reduced dimensions $r_1, \ldots, r_{k-1}, n$.

Now we prove the above classification. If $n-1$ and n are outside of the co-type of P (meaning that $r_k \leq n-2$), we show that P is not special, considering a field L/F admitting a $2n$-dimensional quadratic form q of trivial discriminant and of Witt index $n-2$. Let A be the split central simple L-algebra of degree $2n$ endowed with the quadratic pair given by q. For the corresponding E, the variety E/P has then a rational point, but E is not trivial.

Assume that n is even. If the co-type of P contains precisely one of $n-1, n$ and only even numbers $\leq n-2$, we show that P is not special, considering a field L/F admitting a central simple L-algebra A of degree $2n$ and index 2. We endow such A with a hyperbolic quadratic pair. For the corresponding E, the variety E/P has then a rational point, but E is not trivial.

At this point we proved that a special parabolic P has to satisfy (1) or (2) or (3).

Below we assume that E/P has a rational point for some E over some L/F, A is the corresponding central simple L-algebra with a quadratic pair. If P satisfies (1), the algebra A is necessarily split and the quadratic pair is given by a quadratic form of Witt index $\geq n-1$. Since the discriminant of the quadratic pair (and therefore of the quadratic form) is trivial, the Witt index must be n. Therefore E is trivial and P is special.

If P satisfies (2) and n is even, then A is split again and the Witt index of the quadratic form is $\geq n-1$. For odd n, (2) is a particular case of (3).

Finally, assume that P satisfies (3) (and n is odd). If the co-type of P contains precisely one of $n-1, n$, the algebra A is split as well as the quadratic form. Otherwise, let $P' \supset P$ be the parabolic subgroup of the co-type obtained by removing n from the co-type of P. Since there is a morphism $E/P \to E/P'$ and the variety E/P has a rational point, the variety E/P' also has a rational point, implying that E is split.

Our last case is that of a semispinor group $G = \mathrm{Spin}^{\sim}(2n)$ (here n is at least 4 and even).

- *A parabolic subgroup P of G is special if and only if its co-type satisfies at least one of the following two conditions:*
  (1) contains at least one of the numbers $n-3, n-2, n-1, n$ and at least one odd number $\leq n-3$;
  (2) contains both of the numbers $n-1, n$.

We have a canonical surjection with trivial kernel of the set of isomorphism classes of G-torsors over F onto the set of isomorphism classes of central simple F-algebras of degree $2n$ endowed with a quadratic pair with trivial discriminant and a split component of the Clifford algebra, [15, §29.F].

The interpretation of the flag varieties is the same as in the case of $G = \mathrm{PGO}^{+}(2n)$. And the proof of the classification is a combination of the proofs for $\mathrm{PGO}^{+}(2n)$ and $\mathrm{Spin}(2n)$.

8e. **Exceptional types.** Since the groups of types G_2, F_4, and E_8 are adjoint, [24] suffices to get the classification for them.

For G of type G_2,

- *any parabolic subgroup $P \subsetneq G$ is special.*

For G of type F_4,

- *a parabolic subgroup P of G is special if and only if its co-type contains at least one number different from* 4 (i.e., contains at least one of $1, 2, 3$).

For G of type E_8,

- *a parabolic subgroup P of G is special if and only if its co-type is not contained in* $\{1, 6, 7, 8\}$ (i.e., contains at least one of $2, 3, 4, 5$.)

For E_6 and E_7 we have to distinguish between the adjoint and the simply connected cases.

For adjoint G of type E_6,

- *a parabolic subgroup P of G is special if and only if its co-type is contained neither in $\{1, 6\}$ nor in $\{2, 4\}$.*

For simply connected G of type E_6, we therefore only need to decide which subsets of $\{1, 6\}$ and which subsets of $\{2, 4\}$ are co-types of special parabolic subgroups. It turns out that

- *a parabolic subgroup P of G is special if and only if its co-type is not contained in $\{1, 6\}$.*

Indeed, if the co-type of P is a subset in $\{1,6\}$, the Dynkin diagram of the semisimple part of P is not a union of diagrams of type A or C. Therefore P is not special.

If now the co-type of P is a nonempty subset in $\{2,4\}$, then by [24] for any E over any L/F the variety E/P is generically split if and only if the variety $E/P_{\{2\}}$ is so. By [20, Theorem 5.7(3) and Proposition 4.2] the variety $E/P_{\{2\}}$ is generically split. Therefore P is special.

For adjoint G of type E_7,

- *a parabolic subgroup P of G is special if and only if its co-type is contained neither in $\{1,6,7\}$ nor in $\{1,3,4,6\}$.*

For simply connected G of type E_7, we therefore only need to decide which subsets of $\{1,6,7\}$ and which subsets of $\{1,3,4,6\}$ are co-types of special parabolic subgroups. If the co-type of P is a subset in $\{1,6,7\}$, the Dynkin diagram of the semisimple part of P is not a union of diagrams of type A or C. Therefore P is not special.

As to the second set, the maximal parabolic subgroups $P_{\{3\}}$ and $P_{\{4\}}$ are special by [20, Theorem 5.7(4) and Proposition 4.2]. Therefore

- *a parabolic subgroup P of a simply connected split simple algebraic group G of type E_7 is special if and only if its co-type is not contained in $\{1,6,7\}$ (i.e., contains any of $2,3,4,5$).*

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