BIQUATERNION DIVISION ALGEBRAS OVER RATIONAL FUNCTION FIELDS

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Abstract. Let E be a field which is the center of a quaternion division algebra and which is not real euclidean. Then there exists a biquaternion division algebra over the rational function field $E(t)$ which does not contain any quaternion algebra defined over E. The proof is based on the study of Bezoutian forms developed in [1].

Keywords: Milnor K-theory, quadratic form, valuation, ramification, Bezoutian form

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1. Introduction

Let E be a field of characteristic different from 2. Let $E(t)$ denote the rational function field over E, where t is an indeterminate. It is easy to show that there exists a quaternion division algebra over $E(t)$ if and only if E has some field extension of even degree. The purpose of this article is to provide sufficient conditions to have that there exists a biquaternion division algebra over $E(t)$.

In [6] and [4, Sections 3 and 4] such examples were given in the case where E is a local number field. In this article a different approach will be presented that uses the study of ramification sequences via associated Bezoutian forms developped in [1].

For $a, b \in E^\times$ consider the biquaternion algebra

$$B = \big(t^2 + (a+1)t + a, a\big) \otimes_{E(t)} \big(t^2 + at + a, ab\big)$$

over $E(t)$. We will obtain by Theorem 4.2 that, for having that B is a division algebra, it suffices that the E-quaternion algebra (a, b) is non-split and that $ab, (a-4)b \notin E^{\times 2}$. More precisely, under the same conditions the ramification of B (with respect to the valuations on $E(t)$ which are trivial on E) differs from the ramification of any quaternion algebra over $E(t)$, which means that B has Faddeev index 4, in the terminology of [4].

This will yield the following sufficient condition for the existence of biquaternion division algebras over $E(t)$.

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Theorem. *Assume that E is not real euclidean and that there exists a non-split quaternion algebra over E. Then there exists a biquaternion division algebra over $E(t)$ which does not contain any quaternion algebra defined over E.*

Recall that E is *real euclidean* if the set of squares in E is an ordering of E. It is clear that this case has to be excluded in the Theorem. In fact, if E is real euclidean, then every $E(t)$-quaternion algebra is of the form $(-1, f)$ for some $f \in E[t]$ and it follows that every $E(t)$-biquaternion algebra has zero divisors.

As a consequence of the Theorem one may therefore notice that whenever there exist biquaternion division algebras over $E(t)$ then there exist also biquaternion algebras over $E(t)$ of Faddeev index 4.

Note finally that the converse of the Theorem does not hold. It was shown in [2] that one can construct a field E of cohomological dimension 1 – hence in particular such that every E-quaternion algebra is split – and such that there exist biquaternion division algebras over $E(t)$.

2. Preliminaries

For an E-algebra A and a field extension F/E, we denote by A_F the F-algebra $A \otimes_E F$. Recall that an E-algebra A is *central simple* if and only if $A_F \simeq \mathbb{M}_n(F)$ for some field extension F/E and a positive integer n; we say that A is *split* if one can take $F = E$, that is if $A \simeq \mathbb{M}_n(E)$. Note that any central simple algebra is finite-dimensional and in particular it either has zero divisors or it is a division algebra.

An *E-quaternion algebra* is a 4-dimensional central simple E-algebra. For $a, b \in E^\times$ an E-quaternion algebra denoted $(a, b)_E$ or just (a, b) is obtained by endowing the vector space

$$E \oplus Ei \oplus Ej \oplus Ek$$

with the multiplication given by the rules $i^2 = a, j^2 = b$ and $ij = k = -ji$. Any E-quaternion algebra is isomorphic to $(a, b)_E$ for certain $a, b \in E^\times$. A quaternion algebra is either split or it is a division algebra.

An *E-biquaternion algebra* is an E-algebra which is isomorphic to $Q \otimes_E Q'$ for two E-quaternion algebras Q and Q'. In particular, biquaternion algebras are central simple. Given an E-biquaternion algebra B and an E-quaternion subalgebra Q of B, we can decompose $B \simeq Q \otimes_E Q'$ with the E-quaternion algebra Q' given as the centralizer of Q in B.

For our analysis of quaternion and biquaternion algebras over E and $E(t)$, we will work in the second Milnor K-groups modulo 2 of the fields.

For $n \in \mathbb{N}$ we denote by $\mathsf{k}_n E$ the nth Milnor K-group of E modulo 2; this is the abelian group generated by symbols $\{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n \in E^\times$

which are subject to the defining relations that the map $(E^\times)^n \to \mathsf{k}_n E$ given by $(a_1, \dots, a_n) \mapsto \{a_1, \dots, a_n\}$ is multilinear and further that $\{a_1, \dots, a_n\} = 0$ whenever $a_i \in E^{\times 2}$ for some $i \leqslant n$ or $a_i + a_{i+1} = 1$ for some $i < n$. Note that $\mathsf{k}_0 E \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathsf{k}_1 E \simeq E^\times / E^{\times 2}$. Here we only consider $\mathsf{k}_n E$ (and $\mathsf{k}_n E(t)$) for $n = 1, 2$. The group $\mathsf{k}_2 E$ is in tight relation to the Brauer group.

We denote by $\mathsf{Br}(E)$ the Brauer group of E and by $\mathsf{Br}_2(E)$ its 2-torsion part. Recall that there is a unique homomorphism

$$\mathsf{k}_2 E \to \mathsf{Br}_2(E)$$

that sends any symbol $\{a, b\}$ with $a, b \in E^\times$ to the Brauer equivalence class of the E-quaternion algebra $(a, b)_E$. Merkurjev's Theorem asserts that this is in fact an isomorphism. We only need special instances of this statement. For $a, b \in E^\times$ we have $\{a, b\} = 0$ in $\mathsf{k}_2 E$ if and only if the E-quaternion algebra $(a, b)_E$ is split. Furthermore, for $a, b, c, d \in E^\times$ the E-biquaternion algebra $(a, b) \otimes_E (c, d)$ has zero divisors if and only if $\{a, b\} + \{c, d\} = \{e, f\}$ for certain $e, f \in E^\times$. These two facts can be proven by elementary means, without using Merkurjev's Theorem.

To the study of $\mathsf{k}_2 E(t)$ one uses Milnor's Exact Sequence (2.1). To explain it we first need to define the tame symbol map ∂_v with respect to a $\mathbb{Z}$-valuation v.

Let F be a field. By a *$\mathbb{Z}$-valuation* on F we mean a valuation with value group $\mathbb{Z}$. Given a $\mathbb{Z}$-valuation v on F we denote by $\mathcal{O}_v$ its valuation ring and by κ_v its residue field. For $a \in \mathcal{O}_v$ let $\overline{a}$ denote the natural image of a in κ_v. By [5, (2.1)], for a $\mathbb{Z}$-valuation v on F, there is a unique homomorphism $\partial_v : \mathsf{k}_2 F \to \mathsf{k}_1 \kappa_v$ such that

$$\partial_v(\{f, g\}) = v(f) \cdot \{\overline{g}\} \text{ in } k_1 \kappa_v$$

for $f \in F^\times$ and $g \in \mathcal{O}_v^\times$. For $f, g \in E^\times$ we obtain that $f^{-v(g)} g^{v(f)} \in \mathcal{O}_v^\times$ and

$$\partial_v(\{f, g\}) = \left\{(-1)^{v(f)v(g)} \overline{f^{-v(g)} g^{v(f)}}\right\} \text{ in } \mathsf{k}_1 \kappa_v \,.$$

We turn to the case where $F = E(t)$. Let $\mathcal{P}$ denote the set of monic irreducible polynomials in $E[t]$. Any $p \in \mathcal{P}$ determines a $\mathbb{Z}$-valuation v_p on $E(t)$ which is trivial on E and with $v_p(p) = 1$. There is further a unique $\mathbb{Z}$-valuation v_∞ on $E(t)$ such that $v_\infty(X) = -1$. We set $\mathcal{P}' = \mathcal{P} \cup \{\infty\}$. For $p \in \mathcal{P}'$ we write ∂_p for ∂_{v_p} and we denote by E_p the residue field of v_p. Note that E_p is naturally isomorphic to $E[t]/(p)$ for any $p \in \mathcal{P}$ and that E_∞ is naturally isomorphic to E. We call

$$\partial = \bigoplus_{p \in \mathcal{P}'} \partial_p : \mathsf{k}_2 E(t) \to \bigoplus_{p \in \mathcal{P}'} \mathsf{k}_1 E_p$$

the *ramification map*. For $p \in \mathcal{P}'$, the norm map of the finite extension E_p/E yields a group homomorphism $\mathsf{k}_1 E_p \to \mathsf{k}_1 E$. Summation over these maps for all

$p \in \mathcal{P}'$ yields a homomorphism

$$\mathrm{N} : \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p \to \mathsf{k}_1 E\,.$$

Let $\mathfrak{R}_2(E)$ denote the kernel of N. By [3, (7.2.4) and (7.2.5)] we obtain an exact sequence

$$0 \to \mathsf{k}_2 E \to \mathsf{k}_2 E(t) \xrightarrow{\partial} \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p \xrightarrow{\mathrm{N}} \mathsf{k}_1 E \to 0\,. \tag{2.1}$$

In particular, $\mathfrak{R}_2(E)$ is equal to the image of ∂. The elements of $\mathfrak{R}_2(E)$ are called *ramification sequences*.

For a finite set $S \subseteq \mathcal{P}'$ we call $\sum_{p\in S}[E_p : E]$ the *degree of* S and denote it by $\deg(S)$. For $\rho = (\rho_p)_{p\in\mathcal{P}'} \in \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1 E_p$ we set $\mathsf{Supp}(\rho) = \{p \in \mathcal{P}' \mid \rho_p \neq 0\}$ and abbreviate $\deg(\rho) = \deg(\mathsf{Supp}(\rho))$, and we call this the *support* and the *degree of* ρ. We say that $\rho \in \mathfrak{R}_2(E)$ *is represented by* $\xi \in \mathsf{k}_2 E(t)$ if $\partial(\xi) = \rho$.

3. Bezoutians

We use standard terminology from quadratic form theory. For $n \in \mathbb{N}$ and $a_1, \dots, a_n \in E^\times$ we denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$ by $\langle\!\langle a_1, \dots, a_n\rangle\!\rangle$. The Witt ring of E is denoted by $\mathsf{W}E$. For a nondegenerate quadratic form φ over E we denote by $[\varphi]$ its class in $\mathsf{W}E$ and we set $c\cdot[\varphi] = [c\varphi]$ for $c \in E^\times$. For $c \in E^\times$ we abbreviate $[c] = [\langle c\rangle]$. For $\alpha, \alpha' \in \mathsf{W}E$ we write $\alpha \sim \alpha'$ to indicate that $\alpha' = c\alpha$ for some $c \in E^\times$.

Consider a square-free polynomial $g \in E[t]$ and a polynomial $f \in E[t]$ coprime to g. Let θ denote the class of t in $E_g = E[t]/(g)$, $n = \deg(g)$, and let $s_g : E_g \to E$ be the E-linear form with $s_g(\theta^i) = 0$ for $i = 0, \dots, n-2$ and $s_g(\theta^{n-1}) = 1$. By [1, Proposition 3.1]

$$q : E_g \to E, x \mapsto s_g(f(\theta)x^2)$$

is a nondegenerate quadratic form over E, called the *Bezoutian of* f *modulo* g. We denote the class in $\mathsf{W}E$ given by the Bezoutian of f modulo g by

$$\mathfrak{B}\left(\frac{f}{g}\right)\,.$$

Bezoutians satisfy some computation rules, which are useful.

3.1. **Proposition.** *For $f, g_1, g_2 \in E[t]$ pairwise coprime and with g_1 and g_2 monic and square-free, we have*

$$\mathfrak{B}\left(\frac{f}{g_1 g_2}\right) = \mathfrak{B}\left(\frac{f g_2}{g_1}\right) + \mathfrak{B}\left(\frac{f g_1}{g_2}\right)\,.$$

Proof. See [1, Proposition 3.5]. □

3.2. **Theorem.** *Let $f, g \in E[t]$ be monic, square-free and coprime. Then*

$$\mathfrak{B}\left(\frac{f}{g}\right) + \mathfrak{B}\left(\frac{g}{f}\right) = \begin{cases} 0 & \text{if } \deg(f) \equiv \deg(g) \bmod 2, \\ [1] & \text{if } \deg(f) \not\equiv \deg(g) \bmod 2. \end{cases}$$

Proof. See [1, Theorem 3.8]. □

These two rules will be used without explicit mention in the sequel.

3.3. **Lemma.** *Let a_1, a_2 and $g_1, g_2 \in E[t]$ monic of even degree and relatively coprime and such that $g_1 t$ is a square modulo g_2. Let $f \in E[t]$ be such that $a_i f$ is a square modulo g_i for $i = 1, 2$. Then*

$$\mathfrak{B}\left(\frac{f}{g_1 g_2}\right) \sim [\langle\langle a_1 a_2, g_2(0)\rangle\rangle] .$$

Proof. Let $b = g_2(0)$. As $g_1 t$ is a square modulo g_2, we obtain that

$$\mathfrak{B}\left(\frac{g_1}{g_2}\right) = \mathfrak{B}\left(\frac{t}{g_2}\right) = [1] - \mathfrak{B}\left(\frac{g_2}{t}\right) = [1] - [b] .$$

It follows that

$$\mathfrak{B}\left(\frac{g_2}{g_1}\right) = -\mathfrak{B}\left(\frac{g_1}{g_2}\right) = [b] - [1] .$$

We obtain that

$$\mathfrak{B}\left(\frac{f}{g_1 g_2}\right) = a_1 \mathfrak{B}\left(\frac{g_2}{g_1}\right) + a_2 \mathfrak{B}\left(\frac{g_1}{g_2}\right) = [a_2 \langle\langle a_1 a_2, b\rangle\rangle] .$$

□

4. Ramification sequences not representable by a symbol

In [1, Section 5] Bezoutians are related to ramification sequences and it is shown in [1, Theorem 5.12] that a non-trivial Bezoutian can present an obstruction for the representability of a ramification sequence by a single symbol. Here this will be used to obtain ramification sequences of degree 4 that do not correspond to a symbol.

Given $g \in E[t]$ monic and square-free and $f \in E[t]$ coprime to g, we denote by

$$\mathfrak{R}\left(\frac{f}{g}\right)$$

the element $\rho \in \mathfrak{R}_2(E)$ such that $\rho_p = \{\overline{f}\}$ for all $p \in \mathcal{P}$ dividing g and $\rho_p = 0$ for all other $p \in \mathcal{P}$; note that ρ_∞ is given by the condition that $\rho \in \mathfrak{R}_2(E) = \ker(N)$.

4.1. **Proposition.** *Let $g_1, g_2 \in E[t]$ monic of even degree, coprime and such that $g_1 t$ is a square modulo g_2. Let $a_1, a_2 \in E^\times$ be such that the quadratic form*

$\langle 1, -a_1a_2\rangle$ over $E(t)$ does not represent $g_2(0)$ and for $i = 1, 2$ one has $a_i \notin E_p^{\times 2}$ for any irreducible factor p of g_i. Then

$$\partial\left(\{g_1, a_1\} + \{g_2, a_2\}\right) \neq \partial\left(\sigma\right)$$

for any symbol σ in $\mathsf{k}_2E(t)$.

Proof. We set $\rho = \partial\left(\{g_1, a_1\} + \{g_2, a_2\}\right)$ in $\mathfrak{R}_2(E)$. We have $\rho_\infty = 0$ and $\mathrm{Supp}(\rho) = \{p \in \mathcal{P} \mid p \text{ divides } g_1g_2\}$. In particular $\deg(\rho) = \deg(g_1) + \deg(g_2)$, which is even. Suppose there exists a symbol σ in $\mathsf{k}_2E(t)$ with $\partial(\sigma) = \rho$. It follows by [1, Proposition 4.1] that there exist $f, g, h \in E[t]$ square-free and pairwise coprime and with $g = g_1g_2$ such that $\sigma = \{f, gh\}$, and $\partial(\sigma) = \mathfrak{R}\left(\frac{f}{g}\right)$. By [1, Lemma 4.2] we obtain that

$$\mathfrak{B}\left(\frac{f}{g}\right) = 0\,.$$

Since

$$\mathfrak{R}\left(\frac{f}{g_1}\right) + \mathfrak{R}\left(\frac{f}{g_2}\right) = \mathfrak{R}\left(\frac{f}{g}\right) = \partial(\sigma) = \rho = \mathfrak{R}\left(\frac{a_1}{g_1}\right) + \mathfrak{R}\left(\frac{a_2}{g_2}\right)\,,$$

we have that a_if is a square modulo g_i for $i = 1, 2$. thus by Lemma 3.3 we get that

$$[\langle\langle a_1a_2, g_2(0)\rangle\rangle] \sim \mathfrak{B}\left(\frac{f}{g}\right) = 0\,.$$

Thus $\langle\langle a_1a_2, g_2(0)\rangle\rangle$ is hyperbolic. Hence $g_2(0)$ is represented over E by the quadratic form $\langle 1, -a_1a_2\rangle$, which contradicts the hypothesis. □

We are ready to prove the statements in terms of symbols which were claimed in the introduction in terms of quaternion algebras. For the translation we rely only the fact that the ramification map $\partial : \mathsf{k}_2E(t) \to \bigoplus_{p\in\mathcal{P}'} \mathsf{k}_1E_p$ factors over the natural homomorphism $\mathsf{k}_2E(t) \to \mathsf{Br}_2(E(t))$.

4.2. **Theorem.** *Let $a, b \in E^\times$ be such that $a \notin E^{\times 2}$ and $b \notin aE^{\times 2} \cup (a-4)E^{\times 2}$. The following are equivalent:*

(i) $\{a, b\} = 0$ in k_2E.

(ii) There exists a symbol $\sigma \in \mathsf{k}_2E(t)$ with

$$\partial\left(\left\{t^2 + (a+1)t + a, a\right\} + \left\{t^2 + at + a, ab\right\}\right) = \partial(\sigma)\,.$$

Proof. Set $g_1 = t^2+(a+1)t+a$, $g_2 = t^2+at+a$ and $\rho = \partial\left(\{g_1, a\} + \{g_2, ab\}\right)$. The polynomials g_1 and g_2 are coprime, and we have $g_2(0) = a$ and $g_1t \equiv t^2 \bmod g_2$. The discriminant of g_2 is $a^2 - 4a$. Hence, the conditions on a and b imply that g_2 is separable and ab is a non-square modulo any irreducible factor of g_2. In particular, we have that $\mathrm{Supp}(\rho) = \{p \in \mathcal{P} \mid p \text{ divides } g_1g_2\}$ and $\deg(\rho) = 4$.

Moreover, as $a \notin E^{\times 2}$, we have that a is a non-square modulo the irreducible factors of $g_1 = (t+1)(t+a)$.

If $\{a,b\} \neq 0$, then $\langle 1, -b\rangle$ does not represent $a = g_2(0)$, and we conclude by Proposition 4.1 that $\rho \neq \partial(\sigma)$ for any symbol σ in $\mathsf{k}_2E(t)$. Assume conversely that $\{a,b\} = 0$. Then $\langle\langle a,b\rangle\rangle$ is hyperbolic. Hence choosing $f \in E[t]$ such that $f \equiv a \bmod g_1$ and $f \equiv ab \bmod g_2$, we obtain by Lemma 3.3 that $\mathfrak{B}\left(\frac{f}{g_1g_2}\right) = 0$, and as $\rho = \mathfrak{R}\left(\frac{f}{g_1g_2}\right)$ we conclude by [1, Theorem 5.9] that $\rho = \partial(\sigma)$ for a symbol σ in $\mathsf{k}_2E(t)$. $\square$

Note that $\mathsf{k}_2E = 0$ if and only if every E-quaternion algebra is split. Hence the Theorem in the introduction is covered in the following statement.

4.3. **Theorem.** *Assume that $\mathsf{k}_2E \neq 0$ and that E is not real euclidean. Then we have the following:*

(*a*) *There exists $\rho \in \mathfrak{R}_2(E)$ with $\deg(\rho) = 4$ and such that $\rho \neq \partial(\sigma)$ for any symbol σ in $\mathsf{k}_2E(t)$.*

(*b*) *There exists an $E(t)$-biquaternion division algebra B such that $B \otimes_{E(t)} Q$ is not defined over E for any $E(t)$-quaternion algebra Q. In particular, B does not contain any E-quaternion algebra.*

Proof. We claim that the inclusion

$$yE^{\times 2} \cup (-xy)E^{\times 2} \subseteq xE^{\times 2} \cup (x-4)E^{\times 2}$$

cannot hold for all $x, y \in E^\times$ with $\{x,y\} \neq 0$.

We denote by $\mathsf{D}_E(2)$ the set of nonzero sums of two squares in E and recall that for $c \in E^\times$ we have $\{-1, c\} = 0$ in k_2E if and only if $c \in \mathsf{D}_E(2)$.

If there exists a symbol $\sigma \neq 0$ in k_2E that is not of the form $\{-1,c\}$ for any $c \in E^\times$, then for any $x, y \in E^\times$ with $\{x,y\} = \sigma$, the elements $x, y, -xy$ represent distinct classes in $E^\times/E^{\times 2}$, whereby the above inclusion cannot hold. If there exists $c \in E^\times$ such that $c, -c \in E^\times \smallsetminus \mathsf{D}_E(2)$, then $\{-1,c\} \neq 0$ and for $x = -1$ and $y = c$, the above inclusion does not hold. Suppose now that $\{-1,-1\}$ is the only nonzero symbol in k_2E. Then $E^\times = \mathsf{D}_E(2) \cup -\mathsf{D}_E(2)$ and $-1 \notin \mathsf{D}_E(2)$, in particular E has characteristic zero. With $x = -\frac{9}{4}$ we obtain that $xE^{\times 2} \cup (x-4)E^{\times 2} = -E^{\times 2}$. By the hypothesis, there exists some $c \in \mathsf{D}_E(2) \smallsetminus E^{\times 2}$. We set $y = -c$ and obtain that $\{x,y\} = \{-1,-c\} = \{-1,-1\} \neq 0$ and that the above inclusion does not hold.

Since the preliminary claim is now established, we fix elements $x, y, b \in E^\times$ with $\{x,y\} \neq 0$ and $b \in \left(yE^{\times 2} \cup (-xy)E^{\times 2}\right) \smallsetminus \left(xE^{\times 2} \cup (x-4)E^{\times 2}\right)$. We set $a = x$ and obtain that $\{a,b\} = \{x,y\} \neq 0$ and $b \notin aE^{\times 2} \cup (a-4)E^{\times 2}$. By

Theorem 4.2 the ramification sequence

$$\rho = \partial(\{t^2 + (a+1)t + a, a\} + \{t^2 + at + a, ab\})$$

satisfies the claim in (a).

To show (b), we consider the $E(t)$-biquaternion algebra

$$B = (t^2 + (a+1)t + a, a) \otimes_{E(t)} (t^2 + at + a, ab)\,.$$

For any $f, g \in E(t)^\times$ such that $B \otimes_{E(t)} (f, g)$ can be defined over E, we would obtain that $\rho = \partial(\{f, g\})$, in contradiction to (a). Therefore there exists no $E(t)$-quaternion algebra Q such that $B \otimes_{E(t)} Q$ can be defined over E. In particular, B does not contain any E-quaternion algebra Q', because otherwise the centraliser of $Q'_{E(t)}$ in B would be an $E(t)$-quaternion algebra Q such that $B \otimes_{E(t)} Q$ is defined over E. Since B does in particular not contain $\mathbb{M}_2(E)$, it follows that B is a division algebra. □

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