DEGREE THREE INVARIANTS FOR SEMISIMPLE GROUPS OF TYPES B, C, AND D

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Abstract. We determine the group of reductive cohomological degree 3 invariants of all split semisimple groups of types B, C, and D. We also present a complete description of the cohomological invariants. As an application, we show that the group of degree 3 unramified cohomology of the classifying space BG is trivial for all split semisimple groups G of types B, C, and D.

1. Introduction

A degree d *cohomological invariant* of an algebraic group G defined over a field F is a natural transformation of functors

$$G\textit{-torsors} \to H^d$$

on the category of field extensions over F, where the functor G*-torsors* takes a field K/F to the set $G\textit{-torsors}(K)$ of isomorphism classes of G-torsors over K and the functor H^d takes K to the Galois cohomology $H^d(K) = H^d(K, \mathbb{Q}/\mathbb{Z}(d-1))$. All degree d invariants of G form a group $\operatorname{Inv}^d(G)$. This notion was introduced by Serre, and since then it has been intensively studied by Merkurjev and Rost for $d = 3$ [10, 19].

In this paper, we study degree 3 cohomological invariants of split semisimple groups of Dynkin types B, C, and D. Thus from now on we shall focus on degree 3 invariants. Let G be a split reductive group over a field F. An invariant in $\operatorname{Inv}^3(G)$ is called *normalized* if it vanishes on trivial G-torsors. Such invariants form a subgroup $\operatorname{Inv}^3(G)_{\mathrm{norm}}$ of $\operatorname{Inv}^3(G)$, thus $\operatorname{Inv}^3(G) = \operatorname{Inv}^3(G)_{\mathrm{norm}} \oplus H^3(F)$. A normalized invariant in $\operatorname{Inv}^3(G)_{\mathrm{norm}}$ is called *decomposable* if it is given by a cup product of a degree 2 invariant with a constant invariant of degree 1. The subgroup of decomposable invariants of degree 3 is denoted by $\operatorname{Inv}^3(G)_{\mathrm{dec}}$. The quotient group $\operatorname{Inv}^3(G)_{\mathrm{norm}}/\operatorname{Inv}^3(G)_{\mathrm{dec}}$ is called the group of *indecomposable* invariants and is denoted by $\operatorname{Inv}^3(G)_{\mathrm{ind}}$. This group has been completely determined for all split simple groups in [10], [19], [4] and for some semisimple groups in [17], [1], [2], and [15].

Let G be a split semisimple group over F. A *strict reductive envelope* of G is a split reductive group G_{red} over F such that the derived subgroup of G_{red} is G and the center of G_{red} is a torus. Then, by [18, §10] the restriction map

$$\operatorname{Inv}^3(G_{\mathrm{red}})_{\mathrm{ind}} \to \operatorname{Inv}^3(G)_{\mathrm{ind}}$$

is injective and its image is independent of the choice of a strict reductive envelope G_{red}. This image is called the subgroup of *reductive indecomposable* invariants of G

and is denoted by $\mathrm{Inv}^3(G)_{\mathrm{red}}$. Recently, this subgroup has been completely computed for all split simple groups in [13] and for all split semisimple groups of type A in [17].

In the present paper, we determine the group of reductive indecomposable invariants of all split semisimple groups of types B, C, and D, which completes the cohomological invariants of classical groups. In particular, if each component of the corresponding root system of type B (respectively, type C) has rank at least 2 (respectively, even rank), then the group of indecomposable invariants is also determined as follows (see Theorem 5.1, Theorem 5.5, Theorem 5.6, and Corollary 5.2):

Theorem 1.1. *Let G be an arbitrary split semisimple group of one of the following types: B, C, and D, i.e., $G=(\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ $(n_i\geq 1)$, $(\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ $(n_i\geq 1)$, and $(\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ $(n_i\geq 3)$ respectively for some central subgroup $\boldsymbol{\mu}$ and $m\geq 1$. Let R be the subgroup of Z whose quotient is the character group $\boldsymbol{\mu}^*$, where*

$$Z:=\bigoplus_{i=1}^m Z_i,\ Z_i=\begin{cases}(\mathbb{Z}/2\mathbb{Z})e_i & \text{if } G \text{ is of type } B \text{ or } C,\\ (\mathbb{Z}/4\mathbb{Z})e_i & \text{if } G \text{ is of type } D, n_i \text{ odd},\\ (\mathbb{Z}/2\mathbb{Z})e_{i,1}\bigoplus(\mathbb{Z}/2\mathbb{Z})e_{i,2} & \text{if } G \text{ is of type } D, n_i \text{ even},\end{cases}$$

denotes the character group of the center of the corresponding simply connected semisimple group.

(1) *Assume that G is of type B. Let $l=\dim R$. Then,*

$$\mathrm{Inv}^3(G)_{\mathrm{red}}=(\mathbb{Z}/2\mathbb{Z})^{l-l_1-l_2},$$

where $l_1=\dim\langle e_i\in R\,|\,n_i\leq 2\rangle$, $l_2=\dim\langle e_i+e_j\in R\,|\,e_i,e_j\notin R,\ n_i=n_j=1\rangle$. In particular, if $n_i\geq 2$ for all $1\leq i\leq m$, then

$$\mathrm{Inv}^3(G)_{\mathrm{ind}}=\mathrm{Inv}^3(G)_{\mathrm{red}}=(\mathbb{Z}/2\mathbb{Z})^{l-l_1}.$$

(2) *Assume that G is of type C. Let s denote the number of ranks n_i divisible by 4 and $l=\dim\big(R\cap(\bigoplus_{4\nmid n_i} Z_i)\big)$. Then,*

$$\mathrm{Inv}^3(G)_{\mathrm{red}}=(\mathbb{Z}/2\mathbb{Z})^{s+l-l_1-l_2},$$

where $l_1=\dim\langle e_i\in R\rangle$ and $l_2=\dim\langle e_i+e_j\in R\,|\,e_i,e_j\notin R,\ n_i\equiv n_j\equiv 1 \mod 2\rangle$. In particular, if $n_i\equiv 0 \mod 2$ for all i, then

$$\mathrm{Inv}^3(G)_{\mathrm{ind}}=\mathrm{Inv}^3(G)_{\mathrm{red}}=(\mathbb{Z}/2\mathbb{Z})^{s+l-l_1}.$$

(3) *Assume that G is of type D. Let*

$$\bar{R}=\{(\bar{r}_1,\ldots,\bar{r}_m)\in\bigoplus_{i=1}^m(\mathbb{Z}/2\mathbb{Z})\bar{e}_i\,|\,\sum_{i=1}^m r_i\in R\},\ \text{where } r_i=\begin{cases}2\bar{r}_ie_i & \text{if } n_i \text{ odd},\\ \bar{r}_ie_{i,1}+\bar{r}_ie_{i,2} & \text{if } n_i \text{ even},\end{cases}$$

$R_{1,i}=R\cap Z_i$ for odd n_i, and $R'_{1,i}=R\cap Z_i$ for even n_i. Set

$$R'=\bar{R}\cap\Big(\bigoplus_{4\nmid n_i,R'_{1,i},R_{1,i}\neq Z_i}(\mathbb{Z}/2\mathbb{Z})\bar{e}_i\Big) \text{ with } l=\dim R',\ I_1=\{i\,|\,Z_i=R_{1,i} \text{ or } R'_{1,i}, n_i\neq 3\},$$

$$I_2=\{i\,|\,R'_{1,i}=0,4|n_i\}\cup\{i\,|\,R'_{1,i}=(\mathbb{Z}/2\mathbb{Z})e_{i,1} \text{ or } (\mathbb{Z}/2\mathbb{Z})e_{i,2},n_i\geq 6,4|n_i\} \text{ with } s_i=|I_i|.$$

Then, we have

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{s_1+s_2+l-l_1-l_2}, \ \textit{where}$$

$l_1 = |\{i \mid 4 \nmid n_i, R_{1,i} = 2Z_i \ \textit{or}\ R'_{1,i} = (\mathbb{Z}/2\mathbb{Z})(e_{i,1}+e_{i,2})\}|$, $l_2 = \dim\langle \bar{e}_i + \bar{e}_j \mid R_{1,i} = R_{1,j} = 0,\, 2e_i + 2e_j \in R\rangle$.

For each type of B, C, and D, our main theorem can be restated as follows (see Propositions 6.3, 6.7, 6.13): Assume that F is an algebraically closed field. For type B, let $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$, where $\mathbf{\Gamma}_{2n_i+1}$ is the split even Clifford group [12, §23] and let

$$R \to \mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}}$$

be the homomorphism given by $r \mapsto \mathbf{e}_3(\phi[r])$, where $\phi[r]$ is the quadratic form defined in Remark 6.2 and $\mathbf{e}_3$ denotes the Arason invariant. Then, this morphism is surjective and its kernel is the subspace

$$\langle e_i,\, e_j + e_k \in R \mid e_j, e_k \notin R,\, n_i \le 2,\, n_j = n_k = 1\rangle.$$

For type C, let $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{GSp}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{GSp}_{2n_i}$ is the group of symplectic similitudes [12, §12] and let

$$\bigoplus_{4\,|n_i}(\mathbb{Z}/2\mathbb{Z})e_i \bigoplus \big(R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i)\big) \to \mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}}$$

be the homomorphism given by $e_i \mapsto \Delta_i$ for i such that $4|n_i$ and $r \mapsto \mathbf{e}_3(\phi[r])$ for $r \in R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i)$, where $\phi[r]$ is the quadratic form defined in (59) and Δ_i is the invariant in (60) induced by the Garibaldi-Parimala-Tignol invariant [11]. Then, this morphism is surjective and its kernel is given by

$$\langle e_i,\, e_j + e_k \in R \mid e_j, e_k \notin R,\, n_j \equiv n_k \equiv 1 \mod 2\rangle.$$

For type D, let $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{\Omega}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{\Omega}_{2n_i}$ is the extended Clifford group [12, §13] and let

$$\bigoplus_{i\in I_1\cup I_2}(\mathbb{Z}/2\mathbb{Z})\bar{e}_i \bigoplus R' \to \mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}}$$

be the homomorphism given by $\bar{e}_i \mapsto \mathbf{e}_{3,i}$ for $i \in I_1$, $\bar{e}_i \mapsto \Delta'_i$ for $i \in I_2$, and $r \mapsto \mathbf{e}_3(\phi[r])$ for $r \in R'$, where $\mathbf{e}_{3,i}$ denotes the invariant in (73) induced by the Arason invariant, Δ'_i denotes the invariant in (74) given by the invariant of $\mathbf{PGO}^+_{2n_i}$ (see [19, Theorem 4.7]), and $\phi[r]$ is the quadratic form defined in (59). Then, the morphism is surjective, and its kernel is given by

$$\langle \bar{e}_i,\, \bar{e}_j + \bar{e}_k \in R' \mid \bar{e}_j, \bar{e}_k \notin R',\, n_j \equiv n_k \equiv 1 \mod 2\rangle.$$

Therefore, our main result (Theorem 1.1) tells us that for all split semisimple groups of types B, C, D there are essentially two types of degree three reductive invariants given by the Arason invariant $\mathbf{e}_3$ and the Garibaldi-Parimala-Tignol invariant Δ_i (and its analogue Δ'_i) and no other invariants exist.

An invariant $\alpha \in \mathrm{Inv}^3(G)$ is said to be *unramified* if for any field extension K/F and any element $\eta \in G\text{-}\mathit{torsors}(K)$, its value $\alpha(\eta)$ is contained in $H^3_{\mathrm{nr}}(K)$, where $H^3_{\mathrm{nr}}(K)$ denotes the subgroup in $H^3(K)$ of all unramified elements defined by

$$H^3_{\mathrm{nr}}(K) = \bigcap_v \mathrm{Ker}\left(\partial_v : H^3(K) \to H^2(F(v))\right)$$

for all discrete valuations v on K/F and their residue homomorphisms ∂_v. The subgroup of all unramified invariant in $\mathrm{Inv}^3(G)$ will be denoted by $\mathrm{Inv}^3_{\mathrm{nr}}(G)$. By a theorem of Rost, we have an isomorphism

$$\mathrm{Inv}^3_{\mathrm{nr}}(G) \simeq H^3_{\mathrm{nr}}(F(BG)), \tag{1}$$

where BG is the classifying space of G (see [18], [25]).

A generalized version of Noether's problem asks whether the classifying space BG of an algebraic group G is stably rational or retract rational (see [6], [16]). A way of detecting non-retract rationality is to use unramified cohomology as the following statement: the classifying space BG is not retract rational if there exists a non-constant unramified invariant of degree d for some d [16]. In fact, Saltman gave the first counter example over an algebraically closed field to the original Noether's question by providing certain finite groups which have a non-constant unramified invariant of degree 2 [21]. However, the generalized Noether's problem is still open for a connected algebraic group over an algebraically closed field.

In [5], Bogomolov showed that connected groups have no nontrivial degree 2 unramified invariants, i.e., $\mathrm{Inv}^2_{\mathrm{nr}}(G) = 0$ for a connected group G. In [22] and [23], Saltman showed that the group $\mathrm{Inv}^3_{\mathrm{nr}}(\mathbf{PGL}_n)$ is trivial. Recently, Merkurjev has shown that the group $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is trivial if G is a split simple group [18] or a split semisimple group of type A [14] over an algebraically field F of characteristic 0.

Using the main theorem above we determine the group of unramified invariants of a split semisimple groups of types B, C, and D (see Theorems 6.5, 6.10, 6.15).

Theorem 1.2. *Let* $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ $(n_i \geq 1)$ *or* $(\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ $(n_i \geq 1)$ *or* $(\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ $(n_i \geq 3)$ *defined over an algebraically closed field* F *of characteristic* 0*,* $m \geq 1$*, where* $\boldsymbol{\mu}$ *is an arbitrary central subgroup. Then, there are no nontrivial unramified degree* 3 *invariants for* G*, i.e.,* $\mathrm{Inv}^3_{\mathrm{nr}}(G) = H^3_{\mathrm{nr}}(F(BG)) = 0$.

This paper is organized as follows. In Section 2 we recall some basic definitions and facts used in the rest of the paper. Sections 3-5 are devoted to the computation of the group of degree 3 invariants of a split semisimple group G of types B, C, and D. In the last section, we present a description of the degree 3 invariants of G and a proof of the second main result.

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2. Cohomological invariants of degree 3

In this section we recall some basic notions concerning degree 3 invariants following [10, 19]. We shall frequently use these in the following sections.

2.1. **Invariant quadratic forms.** Let $\tilde{G}$ be a split semisimple simply connected group of Dynkin type $\mathcal{D}$, i.e., $\tilde{G} = G_1 \times \cdots \times G_m$ for some integer $m \geq 1$, where each G_i is a split simple simply connected group of type $\mathcal{D}$. Consider the natural action of the Weyl group $W = W_1 \times \cdots W_m$ of $\tilde{G}$ on the weight lattice $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_m$, where W_i (resp. Λ_i) is the Weyl group (resp. the weight lattice) of G_i. Then, the group of W-invariant quadratic forms $S^2(\Lambda)^W$ on Λ, denoted by $Q(\tilde{G})$, is a sum of cyclic groups

$$Q(\tilde{G}) = \mathbb{Z}q_1 \oplus \cdots \oplus \mathbb{Z}q_m,$$

where q_i is the normalized Killing form of G_i for $1 \leq i \leq m$.

Consider an arbitrary split semisimple group G of Dynkin type $\mathcal{D}$, i.e., $G = \tilde{G}/\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is a central subgroup. Let T be a split maximal torus of G and let T^* be the group of characters of T. Then, the subgroup $Q(G)$ of W-invariant quadratic forms on T^* is given by

$$Q(G) = S^2(T^*) \cap Q(\tilde{G}). \tag{2}$$

2.2. **Degree 3 invariants.** Consider the Chern class map $c_2 : \mathbb{Z}[T^*] \to S^2(T^*)$ defined by $c_2(\sum_i e^{\lambda_i}) = \sum_{i<j} \lambda_i\lambda_j$ [19, §3c], where $\mathbb{Z}[T^*]$ is the group ring of the maximal torus T in Section 2.1 and $\lambda_i \in T^*$. Since $(T^*)^W = 0$, the restriction of c_2 induces a group homomorphism

$$c_2 : \mathbb{Z}[T^*]^W \to Q(G) \tag{3}$$

We shall write $\operatorname{Dec}(G)$ for the image of c_2 in (3). For $\lambda \in T^*$, we denote by $\rho(\lambda) = \sum_{\chi \in W(\lambda)} e^{\chi}$, where $W(\lambda)$ is the W-orbit of λ. Then, the subgroup $\operatorname{Dec}(G)$ is generated by $c_2(\rho(\lambda)) = -\frac{1}{2}\sum_{\chi \in W(\lambda)} \chi^2$. By [19, Theorem 3.9], the indecomposable invariants of G is determined by the following exact sequence

$$0 \to \operatorname{Inv}^3(G)_{\mathrm{dec}} \to \operatorname{Inv}^3(G)_{\mathrm{norm}} \to Q(G)/\operatorname{Dec}(G) \to 0.$$

In particular, if F is algebraically closed, then we have $\operatorname{Inv}^3(G)_{\mathrm{norm}} = Q(G)/\operatorname{Dec}(G)$.

3. The group $Q(G)$ for semisimple groups G of types B, C, D

In the present section, we shall compute the group $Q(G)$ for types B, C, and D.

3.1. **Type B.** Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ be an (arbitrary) split semisimple group of type B, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup for some $k \geq 0$. Let T be the split maximal torus of G (i.e., $T = (\mathbb{G}_m^{\sum n_i})/\boldsymbol{\mu}$) and let

$$R = \{r = (r_1, \ldots, r_m) \in \bigoplus_{i=1}^{m} (\mathbb{Z}/2\mathbb{Z})e_i \mid f_p(r) = 0, 1 \leq p \leq k\} \tag{4}$$

be the subgroup of $\bigoplus_{i=1}^m(\mathbb{Z}/2\mathbb{Z})e_i$ whose quotient is the character group $\boldsymbol{\mu}^*$ for some linear polynomials $f_p \in \mathbb{Z}/2\mathbb{Z}[t_1,\dots,t_m]$. We shall simply write $(\mathbb{Z}/2\mathbb{Z})^m$ for $\bigoplus_{i=1}^m(\mathbb{Z}/2\mathbb{Z})e_i$. Consider the following commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & (\mathbb{Z}/2\mathbb{Z})^m \arrow[r] & \boldsymbol{\mu}^* \arrow[r] & 0 \\ 0 \arrow[r] & T^* \arrow[r] \arrow[u, two heads] & \prod_{i=1}^m \mathbb{Z}^{n_i} \arrow[r] \arrow[u, two heads] & \boldsymbol{\mu}^* \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \tag{5}$$

where T^* is the corresponding character group and the middle map $\prod_{i=1}^m \mathbb{Z}^{n_i} \to (\mathbb{Z}/2\mathbb{Z})^m$ is given by

$$\sum a_{i,j}w_{i,j} \mapsto (\bar{a}_{1,n_1},\dots,\bar{a}_{m,n_m}) \tag{6}$$

for $1\le i\le m$ and $1\le j\le n_i$, where $w_{i,j}$ denote the fundamental weights for the ith component of the root system of G. For the rest of this subsection, we simply write a_i and w_i for a_{i,n_i} and w_{i,n_i}, respectively. Then, it follows from (5) that

$$T^* = \{\sum a_{i,j}w_{i,j} \mid f_p(a_1,\dots,a_m)\equiv 0 \mod 2\}.$$

Let $I=\{1,\dots,m\}$ and let $I_1=\{i\in I \mid f_p(e_i)=0,\, 1\le p\le k\}$, where $\{e_1,\dots,e_m\}$ denotes the standard basis of $\mathbb{Z}^m$. We write the relations $f_p(a_1,\dots,a_m)\equiv 0 \mod 2$ as

$$(a_{i_1},\dots,a_{i_k})^T = B\cdot(a_{j_1},\dots,a_{j_l})^T + (2c_1,\dots,2c_k)^T \tag{7}$$

for some distinct $i_1,\dots,i_k,j_1,\dots,j_l$ such that $\{i_1,\dots,i_k,j_1,\dots,j_l\}=I\backslash I_1$ and some $k\times l$ binary matrix $B=(b_{ij})$ (i.e., $b_{ij}=0$ or 1) with $c_p\in\mathbb{Z}$. Then, we have

$$\sum a_{i,j}w_{i,j} = \sum_{1\le i\le m, 1\le j\le n_i-1} a_{i,j}w_{i,j} + \sum_{i\in I_1} a_iw_i + \sum_{p=1}^k 2c_pw_{i_p} + \sum_{s=1}^l a_{j_s}(w_{j_s}+g_s)$$

where $g_s=(w_{i_1},\dots,w_{i_k})\cdot B_s$ and B_s is the s-th column of B, thus we obtain the following $\mathbb{Z}$-basis of T^*:

$$\{w_{i,j}\}_{1\le i\le m,1\le j\le n_i-1}\cup\{w_i\}_{i\in I_1}\cup\{2w_{i_p}\}_{1\le p\le k}\cup\{w_{j_s}+g_s\}_{1\le s\le l}. \tag{8}$$

Let $v_p=2w_{i_p}$ and $h_p(t_1,\dots,t_l)=b_{p1}t_1+\cdots+b_{pl}t_l\in\mathbb{Z}/2\mathbb{Z}[t_1,\dots,t_l]$ for $1\le p\le k$. Since the group $Q(\tilde{G})$ is generated by the normalized Killing forms

$$q_i = \begin{cases} 2w_i^2 - 2w_{i,n_i-1}w_i - \sum_{j=1}^{n_i-2} w_{i,j}w_{i,j+1} + \sum_{j=1}^{n_i-1} w_{i,j}^2 & \text{if } n_i\ge 1,\\ w_i^2 & \text{if } n_i=1\end{cases}$$

for all $1\le i\le m$, any element of $Q(G)$ is of the form $q=\sum_{i=1}^m d_iq_i$ for some $d_i\in\mathbb{Z}$. Therefore, with respect to the basis (8) we have

$$q = q' + \tfrac{1}{4}\sum_{p=1}^k v_p^2[\delta_{i_p}d_{i_p} + h_p(\delta_{j_1}d_{j_1},\dots,\delta_{j_l}d_{j_l})] + \tfrac{1}{2}\sum_{1\le i<j\le k} v_iv_jh_i(\delta_{j_1}d_{j_1}b_{j_1},\dots,\delta_{j_l}d_{j_l}b_{j_l})$$

for some quadratic form q' with integer coefficients, where

$$\delta_i = \begin{cases} 2 & \text{if } n_i \geq 2 \text{ with } i \in I\backslash I_1, \\ 1 & \text{if } n_i = 1 \text{ with } i \in I\backslash I_1. \end{cases}$$

Hence, by (2) we obtain $q = \sum_{i=1}^m d_i q_i \in Q(G)$ if and only if

$$\delta_{i_p} d_{i_p} + h_p(\delta_{j_1} d_{j_1}, \dots, \delta_{j_l} d_{j_l}) \equiv 0 \mod 4 \tag{9}$$

and

$$h_p(\delta_{j_1} d_{j_1} b_{j_1}, \dots, \delta_{j_l} d_{j_l} b_{j_l}) \equiv 0 \mod 2 \tag{10}$$

for all $1 \leq p \leq k$. In particular, since two systems of equations $\{f_p(t_1, \dots, t_m)\}$ and $\{t_{i_p} + h_p(t_{j_1}, \dots, t_{j_l})\}$ are equivalent we replace the condition (9) by

$$f_p(\delta_1 d_1, \dots, \delta_m d_m) \equiv 0 \mod 4, \tag{11}$$

where we set $\delta_i = 2$ for $i \in I_1$.

Equivalently, we can compute $Q(G)$ with respect to a basis of R as follows. Let

$$R_1 = \langle e_i \,|\, e_i \in R \rangle \text{ and } R_2 = \langle e_i + e_j \,|\, e_i + e_j \in R, e_i, e_j \notin R_1 \rangle \tag{12}$$

be the subspaces of R. We first choose $\{w_i\}_{i \in I_1}$ as a part of basis of T^*. Then, for the remaining part of a basis of T^* we write a given basis of R as

$$(e_{j_1}, \dots, e_{j_l})^T = C(e_{i_1}, \dots, e_{i_k})^T \tag{13}$$

for some $i_1, \dots, i_k, j_1, \dots, j_l$ with $\{i_1, \dots, i_k, j_1, \dots, j_l\} = I\backslash I_1$ and some $l \times k$ binary matrix C such that all basis elements of the form $e_i + e_j$ in R_2 is a part of (13). Then, we have the same $\mathbb{Z}$-basis of T^* as in (8) by replacing g_s in (8) with $g_s = C_s \cdot (w_{i_1}, \dots, w_{i_k})$, where C_s is the s-th row of C. The rest of the computation is the same as in the previous one.

In particular, if either $R = R_1 \oplus R_2$ or $n_i \geq 2$ for all $1 \leq i \leq m$, then the condition (10) becomes trivial, thus

Proposition 3.1. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup for some $k \geq 0$. Let $R = \{r \in (\mathbb{Z}/2\mathbb{Z})^m \,|\, f_p(r) = 0, 1 \leq p \leq k\}$ be the subgroup of $(\boldsymbol{\mu}_2^m)^*$ whose quotient is the character group $\boldsymbol{\mu}^*$ for some linear polynomials $f_j \in \mathbb{Z}/2\mathbb{Z}[t_1, \dots, t_m]$. Assume that either $n_i \geq 2$ for all i or $R = R_1 \oplus R_2$, where R_1 and R_2 are the subgroups of R defined in (12). Then, we have*

$$Q(G) = \{\sum_{i=1}^m d_i q_i \,|\, f_p(\delta_1 d_1, \dots, \delta_m d_m) \equiv 0 \mod 4\}.$$

3.2. **Type C.** Let $G = (\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ be a split semisimple group of type C, where $m, n_i \geq 1$ and $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup for some $k \geq 0$. Let T be the split maximal torus of G and let R be the subgroup of $(\mathbb{Z}/2\mathbb{Z})^m$ as in (4). Then, we have the same commutative diagram (5), replacing the middle vertical map (6) by

$$\sum a_{i,j} e_{i,j} \mapsto (\sum_{j=1}^{n_1} \bar{a}_{1,j}, \dots, \sum_{j=1}^{n_m} \bar{a}_{m,j}),$$

where $e_{i,j}$ denote the standard basis for the ith component of $\prod_{i=1}^{m}\mathbb{Z}^{n_i}$. Then, by (5) we have

$$T^* = \{\sum a_{i,j}e_{i,j} \mid f_p(\sum_{j=1}^{n_1} a_{1,j},\dots,\sum_{j=1}^{n_m} a_{m,j}) \equiv 0 \mod 2\}. \tag{14}$$

We simply write e_i for $e_{i,1}$. Let $e'_{i,j} = e_{i,j} - e_i$ for all $1 \leq i \leq m$ and $2 \leq j \leq n_i$ and let $a_i = \sum_{j=1}^{n_i} a_{i,j}$. Then, we apply the same argument as in type B so that we have the following $\mathbb{Z}$-basis of T^*

$$\{e'_{i,j}\}_{1\leq i\leq m,\, 2\leq j\leq n_i} \cup \{e_i\}_{i\in I_1} \cup \{2e_{i_p}\}_{1\leq p\leq k} \cup \{e_{j_s} + g_s\}_{1\leq s\leq l}, \tag{15}$$

where B is the binary matrix as in (7) and $g_s = (e_{i_1},\dots,e_{i_k})\cdot B_s$.

Let $v_p = 2e_{i_p}$ and let h_p be the polynomial defined as in type B. Since the normalized Killing forms are given by

$$q_i = e_{i,1}^2 + \cdots + e_{i,n_i}^2,$$

for any $q \in Q(G)$ there exist $d_i \in \mathbb{Z}$ such that $q = \sum_{i=1}^{m} d_i q_i$, thus with respect to the basis (15) we have

$$q = q' + \tfrac{1}{4}\sum_{p=1}^{k} v_p^2[n_{i_p}d_{i_p} + h_p(n_{j_1}d_{j_1},\dots,n_{j_l}d_{j_l})] + \tfrac{1}{2}\sum_{1\leq i<j\leq k} v_i v_j h_i(n_{j_1}d_{j_1}b_{j_1},\dots,n_{j_l}d_{j_l}b_{j_l})$$

for some quadratic form q' with integer coefficients. Therefore, by the same argument as in type B we have $q = \sum_{i=1}^{m} d_i q_i \in Q(G)$ if and only if

$$h_p(n_{j_1}d_{j_1}b_{j_1},\dots,n_{j_l}d_{j_l}b_{j_l}) \equiv 0 \mod 2 \text{ and } f_p(\delta_1 n_1 d_1,\dots,\delta_m n_m d_m) \equiv 0 \mod 4 \tag{16}$$

for all $1 \leq p \leq k$, where

$$\delta_i = \begin{cases} 1 & \text{if } i \in I\backslash I_1,\\ \frac{2}{n_i} & \text{if } i \in I_1.\end{cases}$$

Similar to the case of type B, if $R = R_1 \oplus R_2$ or n_i is even for all $1 \leq i \leq m$, then the first condition in (16) becomes obvious, thus

Proposition 3.2. *Let $G = (\prod_{i=1}^{m}\mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup for some $k \geq 0$. Let R, R_1, and R_2 be the groups as in (4) and (12). Assume that either n_i is even for all i or $R = R_1 \oplus R_2$. Then, we have*

$$Q(G) = \{\sum_{i=1}^{m} d_i q_i \mid f_p(\delta_1 n_1 d_1,\dots,\delta_m n_m d_m) \equiv 0 \mod 4\}.$$

3.3. **Type D.** Let $G = (\prod_{i=1}^{m}\mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ be a split semisimple group of type D, $m \geq 1$, $n_i \geq 3$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^{k_1}\times(\boldsymbol{\mu}_4)^{k_2}$ is a subgroup of the center $Z(\prod_{i=1}^{m}\mathbf{Spin}_{2n_i})$ for some $k_1, k_2 \geq 0$. We shall denote the character group $Z(\prod_{i=1}^{m}\mathbf{Spin}_{2n_i})^*$ by

$$Z := \bigoplus_{i=1}^{m} Z_i, \text{ where } Z_i = \begin{cases} (\mathbb{Z}/4\mathbb{Z})e_i & \text{if } n_i \text{ odd},\\ (\mathbb{Z}/2\mathbb{Z})e_{i,1} \bigoplus (\mathbb{Z}/2\mathbb{Z})e_{i,2} & \text{if } n_i \text{ even}.\end{cases} \tag{17}$$

Let T be the split maximal torus of G and let

$$R = \{r \in Z \mid f_p(r) = 0, 1 \le p \le k\}$$

be the subgroup of Z such that $\boldsymbol{\mu}^* \simeq Z/R$ for some linear polynomials $f_1, \dots, f_k \in \mathbb{Z}/4\mathbb{Z}[T_1, \dots, T_m]$ with $k = k_1 + k_2$, where T_i denotes a 2-tuple (t_{i1}, t_{i2}) of variables (resp. a variable t_i) if n_i is even (resp. odd) and the coefficients of t_{i1} and t_{i2} in f_p are either 0 or 2. Then, we have the same diagram (5), replacing the middle vertical map (6) by $\prod_{i=1}^m \mathbb{Z}^{n_i} \to Z$,

$$\sum_{j=1}^{n_i} a_{i,j} w_{i,j} \mapsto A_i := \begin{cases} \big(\overline{a_{i,n_i-1} - a_{i,n_i} + 2S_i}\big) e_i & \text{if } n_i \text{ odd,} \\ \big(\overline{a_{i,n_i-1} + S_i}\big) e_{i1} + \big(\overline{a_{i,n_i} + S_i}\big) e_{i2} & \text{if } n_i \text{ even,} \end{cases} \tag{18}$$

where $S_i = \sum_{j=1}^{[(n_i-1)/2]} a_{i,2j-1}$ and $w_{i,j}$ denote the fundamental weights for the ith component of the root system of G. Therefore, by (5) we have

$$T^* = \{\sum a_{i,j} w_{i,j} \mid f_p(\sum_{i=1}^m A_i) = 0, 1 \le p \le k\}. \tag{19}$$

Let $I_1' = \{i \in I \mid f_p(e_i) = 0 \text{ or } f_p(e_{i,1}) = f_p(e_{i,2}) = 0 \text{ for all } 1 \le p \le k\}$ and $I' = I \backslash I_1'$. In view of the argument in the case of type B we may assume that each relation $f_p(\sum_{i=1}^m A_i) = 0$ can be written as

$$\delta_p a_p = b_p + 4c_p, \text{ where } b_p = \begin{cases} \delta_p a_p + f_p(\sum_{i=1}^m A_i) & \text{if } a_p = a_{i,n_i} \text{ with odd } n_i, \\ \delta_p a_p - f_p(\sum_{i=1}^m A_i) & \text{otherwise,} \end{cases}$$

for some distinct $a_p \in \{a_{i,n_i-1}, a_{i,n_i} \mid i \in I'\}$ with $\delta_p \in \{1, 2\}$ and $c_p \in \mathbb{Z}$ such that the terms $a_1, \dots, a_k$ do not appear in $b_1, \dots, b_k$ and each coefficient of $a_{i,l}$ in b_p is divisible by δ_p.

Let $W_1 = \{w_{i,2j-1} \mid i \in I', 1 \le j \le [(n_i-1)/2]\} \cup \{w_{i,n_i-1}, w_{i,n_i} \mid i \in I'\}$. We simply write $w_p \in W_1$ for w_{i,n_i-1} (resp. w_{i,n_i}) if $a_p = a_{i,n_i-1}$ (resp. $a_p = a_{i,n_i}$). Set

$$g_{i,l} = s_1(i,l) w_1 + \cdots + s_k(i,l) w_k \text{ and } W' = W_1 \backslash \{w_1, \dots, w_k\},$$

where $s_p(i,l)$ denotes the coefficient of $a_{i,l}$ in b_p/δ_p. Then, we obtain the following $\mathbb{Z}$-basis of T^*:

$$\{w_{i,j}\}_{i \in I_1, \forall j} \cup \{w_{i,2j}\}_{i \in I', 1 \le j \le [\frac{n_i-2}{2}]} \cup \{\tfrac{4}{\delta_p} w_p\}_{1 \le p \le k} \cup \{w_{i,l} + g_{i,l}\}_{w_{i,l} \in W'}. \tag{20}$$

Let $v_p = \frac{4}{\delta_p} w_p$ and $v_{i,l} = w_{i,l} + g_{i,l}$. Assume that for each p, w_p is a fundamental weight for the i_p-th component of the root system of G. As the normalized Killing forms are given by

$$q_i = \big(\sum_{j=1}^{n_i} w_{i,j}^2\big) - \big(w_{i,n_i-2} w_{i,n_i} + \sum_{j=1}^{n_i-2} w_{i,j} w_{i,j+1}\big),$$

for any $q \in Q(G)$ there exist $d_i \in \mathbb{Z}$ such that $q = \sum_{i=1}^m d_i q_i$. Hence, with respect to the basis (20) we obtain

$$\begin{aligned} q =& q' + \tfrac{1}{16}\sum_{p=1}^{k} v_p^2\delta_p^2[d_{i_p} + \sum_{w_{i,l}\in W'} d_i s_p(i,l)^2] + \tfrac{1}{8}\sum_{1\le p<u\le k} v_p v_u \delta_p\delta_u[\sum_{w_{i,l}\in W'} d_i s_p(i,l)s_u(i,l)] \\ & - \tfrac{1}{2}\sum_{p=1}^{k} v_p\delta_p[\sum_{w_{i,l}\in W'} v_{il} d_i s_p(i,l)] \end{aligned}$$

for some quadratic form q' with integer coefficients. Hence, $q = \sum_{i=1}^m d_i q_i \in Q(G)$ if and only if

(21)
$$\delta_p^2[d_{i_p} + \sum_{w_{i,l}\in W'} d_i s_p(i,l)^2] \equiv 0 \mod 16, \ \sum_{w_{i,l}\in W'} d_i\delta_p\delta_u s_p(i,l)s_u(i,l) \equiv 0 \mod 8,$$

(22)
$$\text{and } d_i\delta_p s_p(i,l) \equiv 0 \mod 2$$

for all $1 \le p \le k$, $1 \le p < u \le k$, and all (i,l) such that $w_{i,l} \in W'$.

Let $c_{i,1}(p)$, $c_{i,2}(p)$, $c_i(p)$ denote the coefficients of $t_{i,1}, t_{i,2}, t_i$ in f_p, respectively. Note that $c_{i,1}(p)$ and $c_{i,2}(p)$ are either 0 or 2. Since

$$\delta_p^2 + \sum_l \delta_p^2 s_p(i_p,l)^2 = \sum_l \delta_p^2 s_p(i,l)^2 = \begin{cases} 8 & \text{if } c_i(p)=2 \text{ or } c_{i,1}(p)+c_{i,2}(p)=4, \\ 2n_i & \text{if } c_i(p)=\pm 1 \text{ or } c_{i,1}(p)+c_{i,2}(p)=2 \end{cases}$$

for all p and $i \neq i_p$, where the sums range over all l such that $w_{i,l} \in W'$, the first equation in (21) is equivalent to the following equation

$$f_p(T_1,\dots,T_m) \equiv 0 \mod 8, \text{ where } t_i = \begin{cases} \pm n_i d_i & \text{if } c_i(p)=\pm 1, \\ 2d_i & \text{if } c_i(p)=2, \end{cases} \tag{23}$$

$$t_{i,1} = \begin{cases} \frac{n_i d_i}{2} & \text{if } c_{i,1}(p)=2, c_{i,2}(p)=0, \\ d_i & \text{if } c_{i,1}(p)+c_{i,2}(p)=4, \end{cases} \text{ and } t_{i,2} = \begin{cases} \frac{n_i d_i}{2} & \text{if } c_{i,1}(p)=0, c_{i,2}(p)=2, \\ d_i & \text{if } c_{i,1}(p)+c_{i,2}(p)=4 \end{cases}$$

for all $i \in I'$ and we set $t_i = 4d_i$, $t_{i1} = t_{i2} = 2d_i$ for all $i \in I'_1$. Since we have

$$\sum_l s_p(i,l)s_u(i,l) \equiv \begin{cases} \pm 2n_i \mod 8 & \text{if } c_i(p)c_i(u) \equiv \pm 1 \mod 4, \\ 4 \mod 8 & \text{if } c_i(p)c_i(u) \equiv 2 \mod 4, \\ 0 \mod 8 & \text{otherwise} \end{cases}$$

for all $1 \le p < u \le k$ such that $\delta_p = \delta_u = 1$, where the sum ranges over all l such that $w_{i,l} \in W'$, the second equation in (21) is equivalent to

$$\sum_{\{i\in I' | c_i(p)c_i(u)\equiv \pm 1 \mod 4\}} 2d_i + \sum_{\{i\in I' | c_i(p)c_i(u)\equiv 2 \mod 4\}} 4d_i \equiv 0 \mod 8 \tag{24}$$

if $\delta_p = \delta_u = 1$ and

$$4\sum_{i\in I''} d_i \equiv 0 \mod 8$$

for some subset I'' of I' otherwise.

4. The subgroup $\mathrm{Dec}(G)$ for semisimple groups G of types B, C, D

In this section we will compute the subgroup $\mathrm{Dec}(G)$ of decomposable elements of G for types B, C, and D. In this section we shall denote by T and T^* the maximal split torus of G and its character group, respectively and we denote by Λ and Λ_r the weight lattice and the root lattice of G, respectively. The Weyl group of G will be denoted by W.

4.1. **Type B.** Consider a split semisimple group $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ of type B, where $m, n_i \geq 1$ and $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup for some $k \geq 0$. Let $I = \{1, \dots, m\}$.

We first consider the case where G is simply connected (i.e. $G = \tilde{G}$), equivalently $k = 0$. Since

$$\mathrm{Dec}(G_1 \times G_2) = \mathrm{Dec}(G_1) \times \mathrm{Dec}(G_2) \tag{25}$$

for any two semisimple groups G_1 and G_2, it suffices to compute $\mathrm{Dec}(\mathbf{Spin}_{2n+1})$. Observe that $\mathrm{Dec}(\mathbf{Spin}_3) = \mathbb{Z}q$ as $c_2(\rho(w_1)) = -q$ and $\mathrm{Dec}(\mathbf{Spin}_5) = \mathbb{Z}q$ as $c_2(\rho(w_2)) = -q$. Similarly, $c_2(\rho(w_1)) = -2q \in \mathrm{Dec}(\mathbf{Spin}_{2n+1})$ for any $n \geq 2$. As the Weyl group of $\mathbf{Spin}_{2n+1}$ contains a normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ generated by sign switching, we see that $2 \,|\, c_2(\rho(\lambda))$ for any $\lambda \in \Lambda$ (c.f. [10, Part II, §13]), thus $\mathrm{Dec}(\mathbf{Spin}_{2n+1}) = 2\mathbb{Z}q$. Therefore,

$$\mathrm{Dec}(\tilde{G}) = \delta'_1\mathbb{Z}q_1 \oplus \cdots \oplus \delta'_m\mathbb{Z}q_m, \text{ where } \delta'_i = \begin{cases} 2 & \text{if } n_i \geq 3, \\ 1 & \text{if } n_i = 1, 2. \end{cases} \tag{26}$$

Now we assume that G is adjoint (i.e. $G = \bar{G}$), equivalently, $k = m$. Then, $\mathrm{Dec}(\mathbf{O}_3^+) = 4\mathbb{Z}q$ as $c_2(\rho(2w_1)) = -4q$. Similarly, by the same argument as in the simply connected case, we see that $\mathrm{Dec}(\mathbf{O}_{2n+1}^+) = 2\mathbb{Z}q$ for $n \geq 2$ (see [19, Theorem 4.5]). Hence,

$$\mathrm{Dec}(\bar{G}) = \delta''_1\mathbb{Z}q_1 \oplus \cdots \oplus \delta''_m\mathbb{Z}q_m, \text{ where } \delta''_i = \begin{cases} 2 & \text{if } n_i \geq 2, \\ 4 & \text{if } n_i = 1. \end{cases} \tag{27}$$

In general, we show that the subgroup $\mathrm{Dec}(G)$ is determined by certain subgroups of R introduced in Section 3.

Proposition 4.1. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ such that $\boldsymbol{\mu}^* = (\boldsymbol{\mu}_2^m)^*/R$. Let*

$R'_1 = \langle e_i \in R \,|\, n_i \leq 2\rangle$ and $R'_2 = \langle e_i + e_j \in R \,|\, e_i, e_j \notin R,\, n_i = n_j = 1\rangle$ be two subspaces of R with $\dim R'_1 = l_1$ and $\dim R'_2 = l_2$. Then,

$$\mathrm{Dec}(G) = \Big(\bigoplus_{e_i \in R'_1} \mathbb{Z}q_i\Big) \oplus \Big(\bigoplus_{n_i \geq 2,\, e_i \notin R'_1} 2\mathbb{Z}q_i\Big) \oplus \Big(\bigoplus_{r=1}^{l_2} 2\mathbb{Z}q'_r\Big) \oplus \Big(\bigoplus_{s=1}^{l_3} 4\mathbb{Z}q''_s\Big), \tag{28}$$

where $l_3 = m - l_1 - l_2 - |\{i \,|\, n_i \geq 2,\, e_i \notin R'_1\}|$ and q'_r (resp. q''_s) is of the form $q_i + q_j$ (resp. q_i) for some i, j such that $\langle q'_r, q''_s \,|\, 1 \leq r \leq l_2, 1 \leq s \leq l_3\rangle = \langle q_i \,|\, n_i = 1, e_i \notin R'_1\rangle$ over $\mathbb{Z}$.

Proof. It follows from (26) and (27) that we have

$$\delta''_1 \mathbb{Z}q_1 \oplus \cdots \oplus \delta''_m \mathbb{Z}q_m \subseteq \mathrm{Dec}(G) \subseteq \delta'_1 \mathbb{Z}q_1 \oplus \cdots \oplus \delta'_m \mathbb{Z}q_m. \tag{29}$$

By a simple computation, we obtain

$$-c_2(\rho(\chi)) = \begin{cases} a_i^2 q_i & \text{if } \chi = a_i w_{i,1},\ n_i = 1, \\ 2(a_i^2 q_i + a_j^2 q_j) & \text{if } \chi = a_i w_{i,1} + a_j w_{j,1},\ n_i = n_j = 1 \end{cases} \tag{30}$$

for any nonzero integers a_i, a_j and

$$-c_2(\rho(\chi)) = (2a_{i,1}^2 + a_{i,2}^2 + 2a_{i,1}a_{i,2})q_i \ \text{ if } \chi = a_{i,1}w_{i,1} + a_{i,2}w_{i,2},\ n_i = 2 \tag{31}$$

for any integers $a_{i,1}, a_{i,2}$. Let us denote the right hand side of equation (28) by D. We write $D = \bigoplus D_u$, where D_u denotes u-th direct summand of D for $1 \leq u \leq 4$. First, we show that $D \subseteq \mathrm{Dec}(G)$. If $e_i \in R'_1$, then by (8) we have $w_{i,1}, w_{i,2} \in T^*$, thus by (30) and (31) $D_1 \subseteq \mathrm{Dec}(G)$. Similarly, if $e_i + e_j \in R$, then by (8), $w_{i,1} + w_{j,1} \in T^*$, thus by (30) $D_3 \subseteq \mathrm{Dec}(G)$. Finally, it follows from (29) that $D_2 \oplus D_4 \subseteq \mathrm{Dec}(G)$.

On the other hand, a character λ in the weight lattice $\Lambda = \bigoplus_{i=1}^m \Lambda_i$ of G can be written as

$$\lambda = \lambda_{i_1} + \cdots + \lambda_{i_t} = \sum_{j \in J} \lambda_{i_j} + \sum_{j \in K} \lambda_{i_j} \tag{32}$$

for some nonzero characters $\lambda_{i_j} \in \Lambda_{i_j}$ and some subsets $J = \{1 \leq j \leq t \,|\, n_{i_j} = 1\}$ and $K = \{1 \leq j \leq t \,|\, n_{i_j} \geq 2\}$ of I. We show that $c_2(\rho(\lambda)) \in D$ for all $\lambda \in T^*$. First, assume that $t = 1$, i.e., $\lambda = a_{i,1}w_{i,1} + \cdots + a_{i,n_i}w_{i,n_i}$ for some i and $a_{i,1}, \ldots, a_{i,n_i} \in \mathbb{Z}$. If a_{i,n_i} is even, then $\lambda \in (\Lambda_i)_r$, thus by (27) we have $c_2(\rho(\lambda)) \in D_2 \oplus D_4$. Otherwise, as $\lambda \in T^*$ is equivalent to $e_i \in R$, by (26) we get $c_2(\rho(\lambda)) \in D_1 \oplus D_2$.

Now we assume that $t = 2$ and $n_{i_1} = n_{i_2} = 1$, i.e., $\lambda = a_i w_{i,1} + a_j w_{j,1}$ for some i, j and $a_i, a_j \in \mathbb{Z}\backslash\{0\}$ with $n_i = n_j = 1$. If both a_i and a_j are even, then $\lambda \in (\Lambda_i)_r \oplus (\Lambda_j)_r$, so $c_2(\rho(\lambda)) \in D_3 \oplus D_4$. If a_i is even and a_j is odd, then as $\lambda \in T^*$ if and only if $e_j \in R'_1$, we get $c_2(\rho(\lambda)) \in D_1 \oplus D_3 \oplus D_4$. Similarly, if both a_i and a_j are odd, then by (30) we have $c_2(\rho(\lambda)) \in D_3$.

Finally, assume that either $t \geq 3$ or $t = 2$ with $n_{i_1} n_{i_2} \neq 1$. Then, by the action of the normal subgroups $(\mathbb{Z}/2\mathbb{Z})^{n_i}$ of the Weyl group generated by sign switching, we see that the coefficient at each $e_{i_j,l}$ in the expansion of $c_2(\rho(\lambda))$ is divisible by 4 and 2 for $j \in J$ and $j \in K$, respectively, i.e.,

$$c_2(\rho(\lambda)) = 4(\sum_{j\in J} a_j q_j) + 2(\sum_{j\in K} b_j q_j) \tag{33}$$

for some $a_i, b_i \in \mathbb{Z}$. Hence, $c_2(\rho(\lambda)) \in D$, i.e., $\mathrm{Dec}(G) \subseteq D$. □

4.2. **Type C.** Let $G = (\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ be a split semisimple group of type C, $m, n_i \geq 1$, where $\boldsymbol{\mu}$ is a central subgroup. As $c_2(\rho(e_1)) = -q$, we have $\mathrm{Dec}(\mathbf{Sp}_{2n}) = \mathbb{Z}q$. Similarly, as $c_2(\rho(2e_1)) = -4q$ and $c_2(\rho(e_1+e_2)) = -2(n-1)q$, we have $\frac{4}{\gcd(2,n)}q \in \mathrm{Dec}(\mathbf{PGSp}_{2n})$. Moreover, since the Weyl group of $\mathbf{Sp}_{2n}$ contains a normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ generated by sign switching, we see that $\frac{4}{\gcd(2,n)} \mid c_2(\rho(\lambda))$ for any $\lambda \in \Lambda_r$ (c.f. [10, Part II, §14]), thus $\mathrm{Dec}(\mathbf{PGSp}_{2n}) = \frac{4}{\gcd(2,n)}\mathbb{Z}q$ (see [19, §4b]). Therefore, by (25) we have

$$\delta''_1\mathbb{Z}q_1 \oplus \cdots \oplus \delta''_m\mathbb{Z}q_m \subseteq \mathrm{Dec}(G) \subseteq \mathbb{Z}q_1 \oplus \cdots \oplus \mathbb{Z}q_m, \text{ where } \delta''_i = \begin{cases} 4 & \text{if } n_i \text{ odd,} \\ 2 & \text{if } n_i \text{ even.} \end{cases} \tag{34}$$

Similar to the case of type B, we determine the subgroup $\mathrm{Dec}(G)$ for type C.

Proposition 4.2. *Let $G = (\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ such that $\boldsymbol{\mu}^* = (\boldsymbol{\mu}_2^m)^*/R$. Let $R''_2 = \langle e_i + e_j \in R \mid e_i, e_j \notin R,\ n_i \equiv n_j \equiv 1 \mod 2\rangle$ be a subspace of R with $\dim R''_2 = l_2$. Then,*

$$\mathrm{Dec}(G) = (\bigoplus_{e_i\in R} \mathbb{Z}q_i) \oplus (\bigoplus_{n_i\equiv 0 \bmod 2,\, e_i\notin R} 2\mathbb{Z}q_i) \oplus (\bigoplus_{r=1}^{l_2} 2\mathbb{Z}q'_r) \oplus (\bigoplus_{s=1}^{l_3} 4\mathbb{Z}q''_s), \tag{35}$$

where $l_3 = |\{i \mid n_i \equiv 1 \mod 2, e_i \notin R\}| - l_2$ and q'_r (resp. q''_s) is of the form $q_i + q_j$ (resp. q_i) for some i, j such that $\langle q'_r, q''_s \mid 1 \leq r \leq l_2, 1 \leq s \leq l_3\rangle = \langle q_i \mid n_i \equiv 1 \mod 2, e_i \notin R\rangle$ over $\mathbb{Z}$.

Proof. Let T be the split maximal torus of G. Then, by (14) we have

$$T^* = \{\sum a_{i,j}e'_{i,j} + \sum a_i e_{i,1} \mid f_p(a_1, \ldots, a_m) \equiv 0 \mod 2\}, \tag{36}$$

where $e'_{i,j} = e_{i,j} - e_{i,1}$ for all $1 \leq i \leq m$ and $2 \leq j \leq n_i$. First note that we have

$$-c_2(\rho(\chi)) = \begin{cases} a_i^2 q_i & \text{if } \chi \in W(a_i e_{i,1}), \\ 2(n_j a_i^2 q_i + n_i a_j^2 q_j) & \text{if } \chi \in W(a_i e_{i,1} + a_j e_{j,1}) \end{cases} \tag{37}$$

for any nonzero integers a_i and a_j. We shall denote by D the right hand side of equation (35) and write $D = \bigoplus D_u$, where D_u denotes u-th direct summand of D for $1 \leq u \leq 4$. If $e_i \in R$, then by (36) we get $e_{i,1} \in T^*$, thus by (37) $D_1 \subseteq \mathrm{Dec}(G)$. Similarly, by (34) we have $D_2 \oplus D_4 \subseteq \mathrm{Dec}(G)$. Let $e_i + e_j \in R''_2$. Then, by (36) we have $e_{i,1} + e_{j,1} \in T^*$. As both n_i and n_j are odd, by (37) we get $2q_i + 2q_j \in \mathrm{Dec}(G)$, i.e., $D_2 \subseteq \mathrm{Dec}(G)$. Therefore, we get $D \subseteq \mathrm{Dec}(G)$.

Conversely, we shall now show that $c_2(\rho(\lambda)) \in D$ for all $\lambda \in T^*$. Let λ be a character written as in (32) for some subsets

$$J = \{1 \le j \le t \,|\, n_{i_j} \equiv 1 \mod 2\} \text{ and } K = \{1 \le j \le t \,|\, n_{i_j} \equiv 0 \mod 2\} \tag{38}$$

of I. For each $\lambda_i = a_{i,1}e_{i,1} + \cdots + a_{i,n_i}e_{i,n_i} \in \Lambda_i$ we shall denote by $|\lambda_i|$ the number of nonzero coefficients in λ_i. We first assume that $t = 1$, i.e., $\lambda = a_{i,1}w_{i,1} + \cdots + a_{i,n_i}e_{i,n_i}$ for some i and $a_{i,1}, \dots, a_{i,n_i} \in \mathbb{Z}$. Let $a_i = a_{i,1} + \cdots + a_{i,n_i}$. By the same argument as in the proof of Proposition 4.1, we have $c_2(\rho(\lambda)) \in D_2 \oplus D_4$ (resp. $c_2(\rho(\lambda)) \in D_1$) if a_i is even (resp. odd). Now we assume that $t = 2$ with $|\lambda_{i_1}| + |\lambda_{i_2}| = 2$, i.e., $\lambda = a_i e_{i,1} + a_j e_{j,1}$ for some i, j and $a_i, a_j \in \mathbb{Z}\backslash\{0\}$. Then, by the same argument as in the proof of Proposition 4.1 we see from (37) that $c_2(\rho(\lambda)) \in D$.

Assume that either $t \ge 3$ or $t = 2$ with $|\lambda_{i_1}| + |\lambda_{i_2}| \ge 3$. Then, as before it follows from the action of the normal subgroups $(\mathbb{Z}/2\mathbb{Z})^{n_i}$ of W that

$$c_2(\rho(\lambda)) = 4(\sum_{i \in J} a_i q_i) + 2(\sum_{i \in K} b_i q_i)$$

for some $a_i, b_i \in \mathbb{Z}$. Therefore, we get $c_2(\rho(\lambda)) \in D$, thus $\mathrm{Dec}(G) \subseteq D$. □

4.3. **Type** D. Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ be a split semisimple group of type D, where $m \ge 1$, $n_i \ge 3$ and $\boldsymbol{\mu}$ is a central subgroup. Consider the case when G is simple (i.e., $m = 1$ and $n_1 = n$). First of all, as

$$c_2(\rho(\omega_1)) = -2q,\ c_2(\rho(2\omega_1)) = -8q,\ c_2(\rho(\omega_2)) = \begin{cases} -4(n-1)q & \text{if } n \ge 4, \\ -q & \text{if } n = 3, \end{cases} \tag{39}$$

we have $2\mathbb{Z}q \subseteq \mathrm{Dec}(\mathbf{Spin}_{2n})$ for $n \ge 4$, $\mathrm{Dec}(\mathbf{Spin}_6) = \mathbb{Z}q$, $\frac{8}{\gcd(2,n)}\mathbb{Z}q \subseteq \mathrm{Dec}(\mathbf{PGO}^+_{2n})$. On the other hand, as the Weyl group of $\mathbf{Spin}_{2n}$ contains a normal subgroup $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ generated by sign switching of even number of coordinates, we see that $2 \,|\, c_2(\rho(\lambda))$ for any $\lambda \in \Lambda$ with $n \ge 4$ and $\frac{8}{\gcd(2,n)} \,|\, c_2(\rho(\lambda'))$ for all $\lambda' \in \Lambda_r$ with $n \ge 3$ (c.f. [10, Part II, §15]), thus $\mathrm{Dec}(\mathbf{Spin}_{2n}) = 2\mathbb{Z}q$ for any $n \ge 4$ and $\mathrm{Dec}(\mathbf{PGO}^+_{2n}) = \frac{8}{\gcd(2,n)}\mathbb{Z}q$ for any $n \ge 3$ (see [19, §4b]). Hence, by (25) we obtain

$$\delta''_1\mathbb{Z}q_1 \oplus \cdots \oplus \delta''_m\mathbb{Z}q_m \subseteq \mathrm{Dec}(G) \subseteq \delta'_1\mathbb{Z}q_1 \oplus \cdots \oplus \delta'_m\mathbb{Z}q_m, \text{ where} \tag{40}$$

$$\delta''_i = \begin{cases} 8 & \text{if } n_i \text{ odd}, \\ 4 & \text{if } n_i \text{ even}, \end{cases} \text{ and } \delta'_i = \begin{cases} 2 & \text{if } n_i \ge 4, \\ 1 & \text{if } n_i = 3. \end{cases}$$

For the remaining simple groups $\mathbf{O}^+_{2n}$ and $\mathbf{HSpin}_{2n}$ (n even), we also have $2\mathbb{Z}q \subseteq \mathrm{Dec}(\mathbf{O}^+_{2n})$ and $4\mathbb{Z}q \subseteq \mathrm{Dec}(\mathbf{HSpin}_{2n})$ by (39). Moreover, if $n = 4$, then we have

$$c_2(\rho(\omega_3)) = c_2(\rho(\omega_4)) = -2q, \tag{41}$$

thus $2\mathbb{Z}q \subseteq \mathrm{Dec}(\mathbf{HSpin}_8)$. Then, by the action of the Weyl group as above we obtain $\mathrm{Dec}(\mathbf{O}^+_{2n}) = 2\mathbb{Z}q$ for all $n \ge 3$, $\mathrm{Dec}(\mathbf{HSpin}_{2n}) = 4\mathbb{Z}q$ for even $n \ge 6$, and $\mathrm{Dec}(\mathbf{HSpin}_8) = 2\mathbb{Z}q$ ([4, Theorem 5.1]). In general, we determine the subgroup $\mathrm{Dec}(G)$ for type D.

Proposition 4.3. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$, $m \geq 1$, $n_i \geq 3$, where $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of (17) such that $\boldsymbol{\mu}^* = Z/R$, $R_{1,i} = R \cap Z_i$ for odd n_i, and $R'_{1,i} = R \cap Z_i$ for even n_i. Set*

$I'_1 = \{i \mid R_{1,i} \neq 0, n_i \neq 3\} \cup \{i \mid R_{1,i} = 2Z_i, n_i = 3\} \cup \{i \mid R'_{1,i} \neq 0,\, n_i = 4\} \cup$
$\{i \mid e_{i,1} + e_{i,2} \in R'_{1,i},\, n_i \geq 6\},\ I'_2 = \{i \mid R'_{1,i} = 0\} \cup \{i \mid e_{i,1} + e_{i,2} \notin R'_{1,i} \neq 0,\, n_i \geq 6\}.$

Then, we have

$$\mathrm{Dec}(G) = \Big(\bigoplus_{R_{1,i}=Z_i,\, n_i=3} \mathbb{Z}q_i\Big) \oplus \Big(\bigoplus_{i\in I'_1} 2\mathbb{Z}q_i\Big) \oplus \Big(\bigoplus_{i\in I'_2} 4\mathbb{Z}q_i\Big) \oplus \Big(\bigoplus_{r=1}^{l_2} 4\mathbb{Z}q'_r\Big) \oplus \Big(\bigoplus_{s=1}^{l_3} 8\mathbb{Z}q''_s\Big), \tag{42}$$

where $l_2 = \dim_{\mathbb{Z}/2\mathbb{Z}}\langle e_i + e_j \mid 2e_i + 2e_j \in R, R_{1,i} = R_{1,j} = 0\rangle$, $l_3 = |\{i \mid R_{1,i} = 0\}| - l_2$, and q'_r (resp. q''_s) is of the form $q_i + q_j$ (resp. q_i) for some i, j such that $\langle q_i \mid R_{1,i} = 0\rangle = \langle q'_r, q''_s \mid 1 \leq r \leq l_2, 1 \leq s \leq l_3\rangle$ over $\mathbb{Z}$.

Proof. Let $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^{k_1} \times (\boldsymbol{\mu}_4)^{k_2}$ be a central subgroup for some $k_1, k_2 \geq 0$ with $k = k_1 + k_2$. We denote by D the right hand side of equation (42) and we write $D = \bigoplus D_u$, where D_u denotes u-th direct summand of D for $1 \leq u \leq 5$. If $e_i \in R$ with $n_i = 3$, then by (39) $D_1 \subseteq \mathrm{Dec}(G)$. If $2e_i \in R$ or $e_{i,1} + e_{i,2} \in R$, then by (19) we have $w_{i,1} \in T^*$, thus by (39) $2q_i \in \mathrm{Dec}(G)$. Similarly, if $e_{i1} \in R'_{1,i}$ (resp. $e_{i,2} \in R'_{1,i}$) with $n_i = 4$, then by (19) we have $w_{i,3} \in T^*$ (resp. $w_{i,4} \in T^*$), thus by (41) $2q_i \in \mathrm{Dec}(G)$. Therefore, $D_2 \subseteq \mathrm{Dec}(G)$. By a simple calculation, we have

$$-c_2(\rho(\chi)) = 4(n_j a_i^2 q_i + n_i a_j^2 q_j) \text{ if } \chi \in W(a_i w_{i,1} + a_j w_{j,1}) \tag{43}$$

for any nonzero integers a_i and a_j. If $2e_i + 2e_j \in R$ for some $i \neq j$, then again by (19) we obtain $w_{i,1} + w_{j,1} \in T^*$. As both n_i and n_j are odd, by (43) $D_4 \subseteq \mathrm{Dec}(G)$. Finally, it follows by (40) that $D_3 \oplus D_5 \subseteq \mathrm{Dec}(G)$, thus $D \subseteq \mathrm{Dec}(G)$.

Now we prove that $c_2(\rho(\lambda)) \in D$ for all $\lambda \in T^*$. Let λ be a character written as in (32) for some subsets J and K in (38). Assume that $t = 1$, i.e., $\lambda = a_{i,1}w_{i,1} + \cdots + a_{i,n_i}w_{i,n_i}$. Applying the same argument as in the proof of Proposition 4.2 we obtain

$$c_2(\rho(\lambda)) \in \begin{cases} D_4 \oplus D_5 & \text{if } A_i = 0 \text{ with odd } n_i, \\ D_2 & \text{if } A_i \neq 0 \text{ with odd } n_i \geq 5; \text{ or } A_i = 2e_i \text{ with } n_i = 3, \\ D_1 & \text{if } A_i = \pm e_i \text{ with } n_i = 3, \end{cases}$$

where A_i denotes the image of λ in Z as defined in (18) and

$$c_2(\rho(\lambda)) \in \begin{cases} D_3 & \text{if } A_i = 0 \text{ with even } n_i; \text{ or } A_i \neq e_{i,1} + e_{i,2} \text{ with even } n_i \geq 6, \\ D_2 & \text{if } A_i \neq 0 \text{ with } n_i = 4; \text{or } A_i = e_{i,1} + e_{i,2} \text{ with even } n_i \geq 6. \end{cases}$$

We assume that $t = 2$ with λ_{i_1} with $|\lambda_{i_1}| + |\lambda_{i_2}| = 2$. Then, by the same argument as in the proof of Proposition 4.1 together with (43) we get $c_2(\rho(\lambda)) \in D$. Finally, Assume that either $t \geq 3$ or $t = 2$ with $|\lambda_{i_1}| + |\lambda_{i_2}| \geq 3$. Then, by the action of the normal subgroups $(\mathbb{Z}/2\mathbb{Z})^{n_i-1}$ of the Weyl group of G we obtain

$$c_2(\rho(\lambda)) = 8(\sum_{i\in J} a_i q_i) + 4(\sum_{i\in K} b_i q_i)$$

for some $a_i, b_i \in \mathbb{Z}$, thus, $c_2(\rho(\lambda)) \in \mathrm{Dec}(G)$. Hence, $\mathrm{Dec}(G) \subseteq D$. □

5. Degree 3 invariants for semisimple groups G of types B, C, D

We now determine the group of reductive indecomposable invariants of split semisimple groups of types B, C, and D by using the results of Section 3, Propositions 4.1, 4.2, and 4.3.

5.1. Type B.

Theorem 5.1. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\boldsymbol{\mu}^*$. Then,*

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-k-l_1-l_2}, \text{ where}$$

$l_1 = \dim\langle e_i \in R \,|\, n_i \leq 2\rangle$ and $l_2 = \dim\langle e_i + e_j \in R \,|, e_i, e_j \notin R,\, n_i = n_j = 1\rangle$.

Proof. Let $R = \{r = (r_1, \ldots, r_m) \in (\mathbb{Z}/2\mathbb{Z})^m \,|\, f_p(r) = 0, 1 \leq p \leq k\}$ be the subgroup of $(\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\boldsymbol{\mu}^*$ for some linear polynomials $f_p \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m]$. Let $\alpha_{i,j}$ denote the simple roots of the ith component of the root system of G and let $\theta_{i,j}$ be the square of the length of the coroot of $\alpha_{i,j}$. Then, we have

$$\theta_{i,j} = \begin{cases} 2 & \text{if } j = n_i \geq 2, \\ 1 & \text{if } n_i \geq 2, 1 \leq j \leq n_i - 1;\ \text{or } j = n_i = 1. \end{cases}$$

By [13, Proposition 7.1], an indecomposable invariant of G corresponding to $q = \sum_{i=1}^m d_i q_i \in Q(G)$ is reductive indecomposable if and only if the order $|\bar{w}_{i,j}|$ in Λ/T^* divides $\theta_{i,j} d_i$ for all i and j.

Since $|\bar{w}_{i,1}| = 1$ with $n_i = 1$ is equivalent to $e_i \in R$ and

$$|\bar{w}_{i,j}| \leq \begin{cases} 2 & \text{if } j = n_i \geq 2, \\ 1 & \text{if } n_i \geq 2, 1 \leq j \leq n_i - 1, \end{cases}$$

we see that the equation (10) becomes trivial and we may assume that the term $\delta_i d_i (= d_i)$ appears in the equation (11) is divisible by 2. Therefore, any reductive indecomposable invariant of G corresponding to $q = \sum_{i=1}^m d_i q_i \in Q(G)$ satisfies

$$f_p(\frac{\delta_1 d_1}{2}, \ldots, \frac{\delta_m d_m}{2}) \equiv 0 \mod 2, \text{ where } \delta_i = \begin{cases} 2 & \text{if } n_i \geq 2 \text{ or } e_i \in R, \\ 1 & \text{if } n_i = 1 \text{ and } e_i \notin R. \end{cases}$$

for all p, thus we have

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = \frac{\{\sum_{i=1}^m d_i q_i \,|\, f_p(\frac{\delta_1 d_1}{2}, \ldots, \frac{\delta_m d_m}{2}) \equiv 0 \mod 2\}}{\mathrm{Dec}(G)}. \tag{44}$$

Let $R' = R \cap (\bigoplus_{e_i \notin R} (\mathbb{Z}/2\mathbb{Z})e_i)$. Then, the group in the numerator of (44) is generated by

$$\{q_i \,|\, e_i \in R\} \cup \{\sum_{i=1}^{m} (\frac{2r_i}{\delta_i}) q_i \,| r = (r_1, \dots, r_m) \in R'\} \cup \{(\frac{4}{\delta_i}) q_i \,|\, e_i \notin R\}.$$

Hence, the statement for the group of indecomposable reductive invariants follows by Proposition 4.1. □

In particular, under the assumption that the ranks of all components of the root system of G are at least 2 we have the following result.

Corollary 5.2. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\boldsymbol{\mu}^*$. Assume that $n_i \geq 2$ for all $1 \leq i \leq m$. Then,*

$$\mathrm{Inv}^3(G)_{\mathrm{ind}} = \mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-k-l},$$

where $l = \dim\langle e_i \in R \,|\, n_i = 2\rangle$.

Proof. By Theorem 5.1, it suffices to show that $\mathrm{Inv}^3(G)_{\mathrm{ind}} \subseteq \mathrm{Inv}^3(G)_{\mathrm{red}}$. Since $n_i \geq 2$ for all $1 \leq i \leq m$, the inclusion follows directly from the proof of Theorem 5.1. □

Remark 5.3. One can directly compute $\mathrm{Inv}^3(G)_{\mathrm{ind}}$ using Propositions 3.1 and 4.1.

We present below another particular case of Theorem 5.1 (and Theorem 5.5), which follows by the exceptional isomorphism $A_1 = B_1 = C_1$. This result in turn determine the reductive invariants of semisimple groups of type A (see [17, Theorem 7.1]).

Corollary 5.4. *Let $G = (\prod_{i=1}^{m} \mathbf{SL}_2)/\boldsymbol{\mu}$, $m \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\boldsymbol{\mu}^*$. Then,*

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{m-k-l_1-l_2},$$

where $l_1 = \dim\langle e_i \in R\rangle$ and $l_2 = \dim\langle e_i + e_j \in R \,|\, e_i, e_j \notin R\rangle$.

5.2. **Type C.**

Theorem 5.5. *Let $G = (\prod_{i=1}^{m} \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^k$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = \bigoplus_{i=1}^{m} (\mathbb{Z}/2\mathbb{Z})e_i$ whose quotient is the character group $\boldsymbol{\mu}^*$ and let s denote the number of ranks n_i which are divisible by 4. Then,*

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{s+l-l_1-l_2}, \text{ where}$$

$l_1 = \dim\langle e_i \,|\, e_i \in R\rangle$, $l_2 = \dim\langle e_i + e_j \,|\, e_i + e_j \in R,\, e_i, e_j \notin R,\, n_i \equiv n_j \equiv 1 \mod 2\rangle$, and $l = \dim\big(R \cap (\bigoplus_{4 \nmid n_i} (\mathbb{Z}/2\mathbb{Z})e_i)\big)$. In particular, if $n_i \equiv 0 \mod 2$ for all $1 \leq i \leq m$, then

$$\mathrm{Inv}^3(G)_{\mathrm{ind}} = \mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{s+l-l_1}.$$

Proof. We apply arguments similar to the proof of type B. Let θ_{ij} be the square of the length of the coroot corresponding to the simple root of ith component of the root system of G. Then, we have

$$\theta_{i,j} = \begin{cases} 1 & \text{if } j = n_i \geq 1, \\ 2 & \text{otherwise.} \end{cases}$$

Note that $|\bar{w}_{i,n_i}| = 2$ if and only if n_i is odd and the element e_{i,n_i} has order 2 in Λ/T^*. Moreover, by (14) the latter is equivalent to $e_i \notin R$. Hence, by [13, Proposition 7.1] an indecomposable invariant of G corresponding to $q = \sum_{i=1} d_i q_i \in Q(G)$ is reductive indecomposable if and only if $2|d_i$ for all odd n_i such that $e_i \notin R$. Therefore, any reductive indecomposable invariant of G corresponding to $q = \sum_{i=1} d_i q_i \in Q(G)$ obviously satisfies the first equation of (16) and the second equation of (16) divided by 2, i.e.,

$$f_p(\tfrac{\delta_1 n_1 d_1}{2}, \dots, \tfrac{\delta_m n_m d_m}{2}) \equiv 0 \mod 2, \text{ where } \delta_i = \begin{cases} 1 & \text{if } e_i \notin R, \\ \frac{2}{n_i} & \text{if } e_i \in R \end{cases}$$

for all $1 \leq p \leq k$, thus,

$$\operatorname{Inv}^3(G)_{\text{red}} = \{\sum_{i=1}^{m} d_i q_i \,|\, f_p(\tfrac{\delta_1 n_1 d_1}{2}, \dots, \tfrac{\delta_m n_m d_m}{2}) \equiv 0 \mod 2\}/\operatorname{Dec}(G). \tag{45}$$

Let $R' = R \cap (\bigoplus_{4\nmid n_i, e_i \notin R} (\mathbb{Z}/2\mathbb{Z})e_i)$, where R is the subgroup of $\bigoplus_{i=1}^{m} (\mathbb{Z}/2\mathbb{Z})e_i$ as in (4). Then, we easily see that the group in the numerator of (45) is generated by

$$\{q_i \,|\, e_i \in R \text{ or } 4|n_i\} \cup \{\sum_{i=1}^{m} \epsilon_i r_i q_i \,|r = (r_i) \in R'\} \cup \{2\epsilon_i q_i \,|\, e_i \notin R\},\ \epsilon_i = \begin{cases} 1 & \text{if } 2|n_i, \\ 2 & \text{if } 2 \nmid n_i. \end{cases}$$

Hence, the statement immediately follows from Proposition 4.2. If n_i is even for all i, then the same argument together with Proposition 3.2 shows the result for the group of indecomposable invariants. □

5.3. **Type D.**

Theorem 5.6. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$, $m \geq 1$, $n_i \geq 3$, where $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of the character group Z defined in (17) such that $\boldsymbol{\mu}^* = Z/R$, $R_{1,i} = R \cap Z_i$ for odd n_i, $R'_{1,i} = R \cap Z_i$ for even n_i, and let*

$$\bar{R} = \{(\bar{r}_1, \dots, \bar{r}_m) \in \bigoplus_{i=1}^{m} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \,|\, \sum_{i=1}^{m} r_i \in R\}, r_i := \begin{cases} 2\bar{r}_i e_i & \text{if } n_i \text{ odd}, \\ \bar{r}_i e_{i,1} + \bar{r}_i e_{i,2} & \text{if } n_i \text{ even}, \end{cases}$$

$$\text{where } Z := \bigoplus_{i=1}^{m} Z_i \text{ with } Z_i = \begin{cases} (\mathbb{Z}/4\mathbb{Z})e_i & \text{if } n_i \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z})e_{i,1} \bigoplus (\mathbb{Z}/2\mathbb{Z})e_{i,2} & \text{if } n_i \text{ even}. \end{cases}$$

denote the character group of the center of $\prod_{i=1}^m \mathbf{Spin}_{2n_i}$. *Set*

$R' = \bar{R} \cap \big(\bigoplus_{4 \nmid n_i, R'_{1,i}, R_{1,i} \neq Z_i} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \big)$ *with* $l = \dim R'$, $I_1 = \{i \mid Z_i = R_{1,i}$ *or* $R'_{1,i}, n_i \neq 3\}$,

$I_2 = \{i \mid R'_{1,i} = 0, 4|n_i\} \cup \{i \mid R'_{1,i} = (\mathbb{Z}/2\mathbb{Z})e_{i,1}$ *or* $(\mathbb{Z}/2\mathbb{Z})e_{i,2}, n_i \geq 6, 4|n_i\}$ *with* $s_i = |I_i|$. *Then, we have*

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = (\mathbb{Z}/2\mathbb{Z})^{s_1+s_2+l-l_1-l_2}, \text{ where}$$

$l_1 = |\{i \mid 4 \nmid n_i, R_{1,i} = 2Z_i$ *or* $R'_{1i} = (\mathbb{Z}/2\mathbb{Z})(e_{i,1} + e_{i,2})\}|$, *and* $l_2 = \dim\langle \bar{e}_i + \bar{e}_j | 2e_i + 2e_j \in R, R_{1,i} = R_{1,j} = 0\rangle$.

Proof. Let Z denote the character group of the center of $\prod_{i=1}^m \mathbf{Spin}_{2n_i}$ as in (17). Let $\boldsymbol{\mu}$ be a central subgroup such that $\boldsymbol{\mu} \simeq (\boldsymbol{\mu}_2)^{k_1} \times (\boldsymbol{\mu}_4)^{k_2}$ for some $k_1, k_2 \geq 0$ and let $R = \{r \in Z \mid f_p(r) = 0, 1 \leq p \leq k\}$ be the subgroup of Z such that $\boldsymbol{\mu}^* \simeq Z/R$ for some linear polynomials $f_p \in \mathbb{Z}/4\mathbb{Z}[T_1, \ldots, T_m]$ with $k = k_1 + k_2$. We shall use the description of $Q(G)$ in Section 3.3.

Let $\theta_{i,j}$ denote the square of the length of the jth coroot of the ith component of the root system of G. Then, $\theta_{i,j} = 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$. Note that the order of the fundamental weight $w_{i,j}$ in Λ/T^* is trivial for all j if and only if

$$Z_i = \begin{cases} R_{1,i} & \text{if } n_i \text{ odd,} \\ R'_{1,i} & \text{if } n_i \text{ even.} \end{cases}$$

Moreover, if $c_i(p) = \pm 1$ for some $1 \leq p \leq k$, where $c_i(p)$ denotes the coefficient of t_i in f_p, then $R_{1i} = 0$, thus by (19) $2w_{i,n_i} \notin T^*$, i.e., $|\bar{w}_{i,n_i}| = 4$. Hence, by [13, Proposition 7.1] any reductive indecomposable invariant of G corresponding to $q = \sum_{i=1} d_i q_i \in Q(G)$ satisfies (22) and (24). Therefore, it follows by (23) that

$$\mathrm{Inv}^3(G)_{\mathrm{red}} = \frac{\{\sum_{i=1}^m d_i q_i \mid \bar{f}_p(\epsilon_1 d_1, \cdots, \epsilon_m d_m) \equiv 0 \mod 2\}}{\mathrm{Dec}(G)} \tag{46}$$

where, $\bar{f}_p \in \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m]$ denotes the image of f_p under the following map

$$\mathbb{Z}/4\mathbb{Z}[T] \to \mathbb{Z}/4\mathbb{Z}[t_1, \ldots, t_m] \to \mathbb{Z}/2\mathbb{Z}[t_1, \ldots, t_m] \text{ given by } 2t_{i1}, 2t_{i2} \mapsto t_i, t_i \mapsto t_i$$

$$\text{and } \epsilon_i = \begin{cases} 1 & \text{if } Z_i = R_{1,i} \text{ or } R'_{1,i}, \\ \frac{1}{2} & \text{if } c_i(p) = 2 \text{ or } c_{i1}(p) + c_{i2}(p) = 4, \\ \frac{n_i}{4} & \text{otherwise.} \end{cases}$$

Let $\bar{R} = \{\bar{r} = (\bar{r}_1, \ldots, \bar{r}_m) \in \bigoplus_{i=1}^m (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \mid \bar{f}_p(\bar{r}) \equiv 0 \mod 2\}$, equivalently

$$\bar{R} = \{(\bar{r}_1, \ldots, \bar{r}_m) \in (\mathbb{Z}/2\mathbb{Z})^m \mid \sum_{i=1}^m r_i \in R\}, \text{ where } r_i := \begin{cases} 2\bar{r}_i e_i & \text{if } n_i \text{ odd,} \\ \bar{r}_i e_{i,1} + \bar{r}_i e_{i,2} & \text{if } n_i \text{ even} \end{cases}$$

and let $R' = \bar{R} \cap \big(\bigoplus_{4 \nmid n_i, R'_{1,i}, R_{1,i} \neq Z_i} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \big)$. Observe that $\bar{f}_p(\bar{e}_i) \equiv 0 \mod 2$ for all p with n_i odd if and only if either $c_i(p) = 0$ or 2 for all p (i.e., $f_p(e_i) \equiv 0$ or $f_p(2e_i) \equiv 0 \mod 4$, respectively) and this, in turn, is equivalent to $R_{1,i} = Z_i$ or $2Z_i$. Similarly, $\bar{f}_p(\bar{e}_i) \equiv 0 \mod 2$ for all p with n_i even if and only if either $c_{i1}(p) = c_{i2}(p) = 0$

or $c_{i1}(p) = c_{i2}(p) = 2$ for all p (i.e., $f_p(e_{i1}) \equiv f_p(e_{i2}) \equiv 0$ or $f_p(e_{i1} + e_{i2}) \equiv 0 \mod 4$, respectively) and this, in turn, is equivalent to $R'_{1,i} = Z_i$ or $(\mathbb{Z}/2\mathbb{Z})(e_{i1} + e_{i2})$. Therefore, we see that the group in the numerator of (46) is generated by

$$\{2q_i \mid R_{1,i} = 2Z_i \text{ or } R'_{1,i} = (\mathbb{Z}/2\mathbb{Z})(e_{i1} + e_{i2}) \text{ or } e_{i,1} + e_{i,2} \notin R'_{1,i}, 4|n_i\} \cup \{8q_i \mid R_{1,i} = 0\}$$
$$\cup \{q_i \mid Z_i = R_{1,i} \text{ or } R'_{1,i}\} \cup \{4q_i \mid e_{i,1} + e_{i,2} \notin R'_{1,i}, 4 \nmid n_i\} \cup \{\sum_{i=1}^{m} \epsilon'_i r'_i q_i\}$$

for all $r' = (r'_1, \dots, r'_m) \in R' \backslash \{\bar{e}_i \mid 1 \le i \le m\}$, where $\epsilon'_i = 2$ (resp. 4) if n_i is even (resp. odd). Therefore, the statement immediately follows by Proposition 4.3. □

6. Unramified invariants for semisimple groups G of types B, C, D

In this section, we first describe torsors for the corresponding reductive groups in Lemmas 6.1, 6.6, and 6.11. Then, using this together with Theorems 5.1, 5.5, and 5.6, we present a complete description of the corresponding cohomological invariants in Propositions 6.3, 6.7, and 6.13. Finally, using such descriptions, we show that there are no nontrivial unramified degree 3 invariants for semisimple groups of types B, C, and D (see Theorems 6.5, 6.10, 6.15). In this section, we assume that the base field F is of characteristic 0. We shall write $\langle a_1, \dots, a_n\rangle = \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle$ for the diagonal quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$ and write $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ for the n-fold Pfister form.

6.1. **Type B.**

Lemma 6.1. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$, $m, n_i \ge 1$, where $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\boldsymbol{\mu}^*$. Set $G_{\mathrm{red}} = (\prod_{i=1}^{m} \boldsymbol{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$, where $\boldsymbol{\Gamma}_{2n_i+1}$ is the split even Clifford group. Then, for any field extension K/F the first Galois cohomology set $H^1(K, G_{\mathrm{red}})$ is bijective to the set of m-tuples of quadratic forms $(\phi_1, \dots, \phi_m)$ with $\dim\phi_i = 2n_i + 1$, $\operatorname{disc}\phi_i = 1$ such that for all $r = e_{i_1} + \cdots + e_{i_s} \in R$, $i_1 < \cdots < i_s$,*

$$I^3(K) \ni \begin{cases} \perp_{p=1}^{s} (-1)^p\phi_{i_p} & \text{if } s \text{ is even,} \\ (\perp_{p=1}^{s} (-1)^p\phi_{i_p}) \perp \langle 1\rangle & \text{otherwise,} \end{cases} \tag{47}$$

where $\operatorname{disc}\phi_i$ denotes the discriminant of ϕ_i and $I^3(K)$ denotes the cubic power of the fundamental ideal $I(K)$ in the Witt ring of K.

Proof. Let $G_{\mathrm{red}} = (\prod_{i=1}^{m} \boldsymbol{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$, where $\boldsymbol{\Gamma}_{2n_i+1}$ denotes the split even Clifford group. Consider the natural exact sequence

$$1 \to (\mathbb{G}_m)^m/\boldsymbol{\mu} \to G_{\mathrm{red}} \to \prod_{i=1}^{m} \mathbf{O}^+_{2n_i+1} \to 1.$$

Then, by Hilbert Theorem 90 and [24, Proposition 42], this sequence yields a bijection between the set $H^1(F, G_{\mathrm{red}})$ and the kernel of the connecting map which factors as

$$H^1(F, \prod_{i=1}^{m} \mathbf{O}^+_{2n_i+1}) \to H^2(F, (\boldsymbol{\mu}_2)^m) = \mathrm{Br}_2(F)^m \xrightarrow{\tau} H^2(F, (\boldsymbol{\mu}_2)^m/\boldsymbol{\mu}),$$

where the first map sends an m-tuple of quadratic forms $(\phi_1, \ldots, \phi_m)$ with $\dim \phi_i = 2n_i + 1$, $\mathrm{disc}(\phi_i) = 1$ to the m-tuple $(C_0(\phi_1), \ldots, C_0(\phi_m))$ of even Clifford algebras $C_0(\phi_i)$ associated to ϕ_i and the map τ is induced by the natural surjection $(\boldsymbol{\mu}_2)^m \to (\boldsymbol{\mu}_2)^m/\boldsymbol{\mu}$. Since $(C_0(\phi_1), \ldots, C_0(\phi_m)) \in \mathrm{Ker}(\tau)$ if and only if it is contained in the kernel of the composition

$$H^2(F, (\boldsymbol{\mu}_2)^m) \xrightarrow{\tau} H^2(F, (\boldsymbol{\mu}_2)^m/\boldsymbol{\mu}) \xrightarrow{r_*} H^2(F, \mathbb{G}_m) \tag{48}$$

for all $r \in R = ((\boldsymbol{\mu}_2)^m/\boldsymbol{\mu})^*$, we have

$$H^1(F, G_{\mathrm{red}}) \simeq \{(\phi_1, \ldots, \phi_m) \mid \dim \phi_i = 2n_i + 1, \mathrm{disc}\, \phi_i = 1, \sum_{i=1}^{m} r_i C_0(\phi_i) = 0\} \tag{49}$$

for all $r = (r_i) \in R$.

Write an element $r \in R$ as $r = e_{i_1} + \cdots + e_{i_s}$ for some $i_1 < \cdots < i_s$, so that the condition $\sum_{i=1}^m r_i C_0(\phi_i) = 0$ in (49) is equal to $\sum_{p=1}^s C_0(\phi_{i_p}) = 0$. Assume that s is even. Since $\mathrm{disc}(-\phi_{i_p} \perp \phi_{i_{p+1}}) = 1$ for any $1 \leq p \leq s/2$,

$$C_0(\psi) = C_0(-\psi), \text{ and } C_0(\phi) + C_0(\phi') = C(\phi \perp \phi')$$

for any quadratic form ψ and any odd-dimensional quadratic forms ϕ and ϕ', where $C(\phi \perp \phi')$ is the corresponding Clifford algebra, we have

$$0 = \sum_{p=1}^{s} C_0(\phi_{i_p}) = C(-\phi_{i_1} \perp \phi_{i_2} \perp \cdots \perp -\phi_{i_{s-1}} \perp \phi_{i_s}),$$

which is equivalent to $(-\phi_{i_1} \perp \phi_{i_2}) \perp \cdots \perp (-\phi_{i_{s-1}} \perp \phi_{i_s}) \in I^3(F)$ by [9, Theorem 14.3]. Now we assume that s is odd. Since $C_0(\phi \perp \langle 1 \rangle) = C_0(\phi)$ for any odd-dimensional quadratic form ϕ and $\mathrm{disc}(-\phi_{i_s} \perp \langle 1 \rangle) = 1$, the same argument shows that $(-\phi_{i_1} \perp \phi_{i_2}) \perp \cdots \perp (-\phi_{i_{s-2}} \perp \phi_{i_{s-1}}) \perp (-\phi_{i_s} \perp \langle 1 \rangle) \in I^3(F)$. □

Remark 6.2. If we assume that $-1 \in (F^\times)^2$, then the condition (47) in Lemma 6.1 can be simplified without sign changes as follows:

$$H^1(K, G_{\mathrm{red}}) \simeq \{\phi := (\phi_1, \ldots, \phi_m) \mid \dim \phi_i = 2n_i + 1, \mathrm{disc}\, \phi_i = 1, \phi[r] \in I^3(K)\}$$

for all $r = (r_i) \in R$, where

$$\phi[r] := \begin{cases} \perp_{i=1}^m r_i \phi_i & \text{if } \sum_{i=1}^m r_i \equiv 0 \mod 2, \\ (\perp_{i=1}^m r_i \phi_i) \perp \langle 1 \rangle & \text{otherwise.} \end{cases}$$

Proposition 6.3. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ defined over an algebraically closed field F, where $m, n_i \geq 1$, $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^*$ whose quotient is the character group $\boldsymbol{\mu}^*$. Set $G_{\mathrm{red}} = (\prod_{i=1}^m \boldsymbol{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$, where $\boldsymbol{\Gamma}_{2n_i+1}$*

is the split even Clifford group. Then, every normalized invariant in $\mathrm{Inv}^3(G_{\mathrm{red}})$ *is of the form* $\mathbf{e}_3(\phi[r])$ *for some* $r \in R$*, where* $\phi[r]$ *is the quadratic form defined in Remark 6.2 and* $\mathbf{e}_3 : I^3(K) \to H^3(K)$ *denotes the Arason invariant over a field extension* K/F*. Moreover, we have*

$$\mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}} \simeq \frac{R}{\langle e_i,\, e_j + e_k \in R \,|\, e_j, e_k \notin R,\, n_i \leq 2,\, n_j = n_k = 1 \rangle}. \tag{50}$$

Proof. Observe that $\mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}} = \mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{ind}}$ as F is algebraically closed. Since $\phi[r] \in I^3(K)$ for any $r \in R$, the Arason invariant gives a normalized invariant of G_{red} of order dividing 2 that sends an m-tuple $\phi \in H^1(K, G_{\mathrm{red}})$ to $\mathbf{e}_3(\phi[r]) \in H^3(K)$.

Let $r \in R_1' + R_2'$, where R_1' and R_2' denote the subgroups of R defined in Proposition 4.1. Then, as every 4 and 6-dimensional quadratic forms in $I^3(K)$ are hyperbolic, the invariant $\mathbf{e}_3(\phi[r])$ vanishes.

Now we show that the invariant $\mathbf{e}_3(\phi[r])$ is nontrivial for any $r \in R \backslash (R_1' + R_2')$. Let $G_{\mathrm{red}}' = (\mathbf{\Gamma}_3)^m/\boldsymbol{\mu}$. If R is a subgroup such that every element r in R has at least 3 nonzero components, then by [14, Lemma 4.3] and the exceptional isomorphism $A_1 = B_1$, any invariant of G_{red}' is nontrivial. Hence, it follows from the map

$$\mathrm{Inv}^3(G_{\mathrm{red}}) \to \mathrm{Inv}^3(G_{\mathrm{red}}')$$

induced by the standard embedding $\mathbf{\Gamma}_3 \to \mathbf{\Gamma}_{2n_i+1}$ that every invariant $\mathbf{e}_3(\phi[r])$ is nontrivial. Otherwise, by the proof of Lemma 6.4 each invariant $\mathbf{e}_3(\phi[r])$ is nontrivial, thus the statements follow from Theorem 5.1. □

Recall from Section 3 the following subgroups of R.

$$R_1 = \langle e_i \in R \rangle \text{ and } R_2 = \langle e_i + e_j \in R \,|\, e_i, e_j \notin R_1 \rangle.$$

We shall need the following key lemma.

Lemma 6.4. *Let* $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ *defined over an algebraically closed field* F*, where* $m, n_i \geq 1$*,* $\boldsymbol{\mu}$ *is a central subgroup. Set* $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$*. Then, every normalized invariant in* $\mathrm{Inv}^3(G_{\mathrm{red}})$ *is ramified if either* $n_i \geq 3$ *for some* i *with* $e_i \in R_1$ *or* $n_j + n_k \geq 3$ *for some* j *and* k *such that* $e_j + e_k \in R_2$*.*

Proof. Let R_3 be a complementary subspace of $R_1 + R_2$ in R. Then, by Proposition 6.3 any normalized invariant α in $\mathrm{Inv}^3(G_{\mathrm{red}})$ can be written as

$$\alpha(\phi) = \mathbf{e}_3(\phi[r_1]) + \mathbf{e}_3(\phi[r_2]) + \mathbf{e}_3(\phi[r_3])$$

for some $r_i \in R_i$, $1 \leq i \leq 3$, where $\phi = (\phi_1, \ldots, \phi_m)$ denotes a G_{red}-torsor.

Suppose that $r_1 \in R_1$ is nonzero. Then, we may assume that $r_1 = e_1$ with $n_1 \geq 3$. Choose a division quaternion algebra (x, y) over a field extension K/F. Find $\phi[e_1] = \phi_1$ such that $\phi_1 \perp \langle 1 \rangle = \langle\langle x, y, z \rangle\rangle \perp h$ over the field of formal Laurent series $K((z))$ and set $\phi_i = h \perp \langle 1 \rangle$ for all $2 \leq i \leq m$, where h denotes a hyperbolic form. Then, we have $\partial_z(\alpha(\phi)) = (x, y) \neq 0$, where ∂_z denotes the residue map, thus $\alpha(\phi)$ ramifies.

Now we may assume that $\alpha(\phi) = \mathbf{e}_3(\phi[r_2]) + \mathbf{e}_3(\phi[r_3])$ with $r_2 \neq 0$. To show that $\alpha(\phi)$ ramifies, we shall choose bases of R_2 and a complementary subspace of R_2. For simplicity, we will write $e(i_1, \ldots, i_k)$ for $e_{i_1} + \cdots + e_{i_k}$. We first select $e(i_p, i_{p,q}) \in R_2$, where i_p, $i_{p,q}$ ($1 \leq p \leq k$, $1 \leq q \leq m_p$) are all distinct integers for some $m_1, \dots, m_k$, so that $B_2 := \{e(i_p, i_{p,q})\}$ is a basis of R_2. In particular, if $n_{i_{p,q}} = 1$ for some p and q, say $n_{i_{1,1}} = 1$, then we replace the subset $\{e(i_1, i_{1,q}) \,|\, 1 \leq q \leq m_1\}$ of B_2 by $\{e(i_{1,1}, i_1), e(i_{1,1}, i_{1q}) \,|\, 2 \leq q \leq m_1\}$ so that we may assume that $n_{i_p} = 1$. We set

$$I_2 = \{i_p \,|\, 1 \leq p \leq k\} \text{ and } I_2' = \{i_p, i_{p,q} \,|\, 1 \leq p \leq k, 1 \leq q \leq m_p\}.$$

We select a basis B_3 of a complementary subspace of R_2. First, we find any basis D_3 of R_3. Then, we modify each element d of D_3 by adding $e(i_p, i_{p,q})$ to it whenever either $e(i_{p,q})$ or $e(i_p, i_{p,q})$ appears in d. Hence, we obtain a basis $C_3 := \{e(k_1, \dots, k_l)\}$ of a complementary subspace of R_2 such that the intersection

$$\big(\bigcup\{k_1, \dots, k_l \,|\, e(k_1, \dots, k_l) \in C_3\}\big) \cap I_2',$$

where the union is over all elements of C_3, is a subset of I_2. We denote by J_2 the intersection. We can divide all elements of the basis C_3 into two types: either $e(i_p)$ for some $i_p \in J_2$ appears in $e(k_1, \dots, k_l) \in C_3$ (the first type) or not (the second type).

We first select basis elements from the first type elements as follows. We choose any element $b(i_1)$ in C_3 of the first type such that $e(i_1)$ appears in the element (if there is no element of the first type, we skip the selection of elements of the first type). We write $b(i_1) := e(i_1) + b'(i_1)$, where $e(i_1)$ does not appear in $b'(i_1)$. We modify every element of the first type by adding $b(i_1)$ to the element whenever $e(i_1)$ appears in the element. For simplicity, we shall use the same notation C_3 for the modified basis of C_3. Then, $e(i_1)$ appears only in $b(i_1)$ among the elements of C_3. Now we choose another element $b(i_2)$ of the first type in which $e(i_2)$ appears for some $i_2 \in J_2$. We write $b(i_2) := e(i_2) + b'(i_2)$, where $e(i_2)$ does not appear in $b'(i_2)$. As $e(i_1)$ appears only in $b(i_1)$, both $e(i_1)$ and $e(i_2)$ do not appear in $b'(i_2)$. Again, we modify every element of the first type by adding $b(i_2)$ to the element whenever $e(i_2)$ appears in the element. In particular, both $e(i_1)$ and $e(i_2)$ do not appear in the modified $b'(i_1)$. We do the same procedure successively for all elements of the first type so that we have chosen basis elements $b(i_p) := e(i_p) + b'(i_p)$ for all i_p in some subset $J_2' \subseteq J_2$ such that all the terms $e(i_p)$ do not appear in $b'(i_p)$.

Similarly, we select basis elements from the second type elements. We choose any element $b(j_1)$ of the second type with $j_1 \notin J_2$, so that we write $b(j_1) := e(j_1) + b'(j_1)$, where $e(j_1)$ does not appear in $b'(j_1)$. We modify every element of C_3 (i.e., $b(i_p)$ and elements of the second type) by adding $b(j_1)$ to the element whenever $e(j_1)$ appears in the element. Then, in particular, all the terms $e(i_p)$ and $e(j_1)$ do not appear in the modified $b'(i_p)$. Now we choose another element $b(j_2)$ of the second type for some $j_2 \notin J_2$, so that we have $b(j_2) := e(j_2) + b'(j_2)$, where both $e(j_1)$ and $e(j_2)$ do not appear in $b'(j_2)$. Again we modify every element of C_3 by adding $b(j_2)$ to the element whenever $e(j_2)$ appears in the element. Then, both $e(j_1)$ and $e(j_2)$ do not appear in modified $b'(j_2)$ and all the terms $e(i_p)$, $e(j_1)$, and $e(j_2)$ do not appear in the modified $b'(i_p)$. Applying the same procedure to all elements of the second type, we obtain the following basis B_3 of a complementary subspace of R_2:

$$b(i_p) := e(i_p) + b'(i_p),\ b(j_1) := e(j_1) + b'(j_1),\ \cdots,\ b(j_s) := e(j_s) + b'(j_s)$$

for all $i_p \in J_2'$ and some distinct $j_1, \dots, j_s \notin J_2$ such that all the terms $e(i_p)$ and $e(j_r)$ do not appear in $b'(i_p), b'(j_r)$ for all $1 \leq r \leq s$, thus

$$B_3 = \{b(i_p), b(j_r) \mid i_p \in J_2', 1 \leq r \leq s\}.$$

Using the basis $B_2 \cup B_3$, we rewrite the invariant $\alpha(\phi) = \mathbf{e}_3(\phi[r_2]) + \mathbf{e}_3(\phi[r_3])$ as

$$\alpha(\phi) = \sum_{b \in B_2'} \mathbf{e}_3(\phi[b]) + \sum_{b \in B_3'} \mathbf{e}_3(\phi[b]) \tag{51}$$

for some subsets $\emptyset \neq B_2' \subseteq B_2$ and $B_3' \subseteq B_3$. Now we show that the invariant $\alpha(\phi)$ in (51) ramifies. It is convenient to split the proof into two cases.

Case 1: $\exists\, e(i_p, i_{p,q}) \in B_2'$ with $n_{i_p} + n_{i_{p,q}} \geq 3$ such that $i_p \notin J_2'$. Let $e(i_u, i_{u,v}) \in B_2'$ be such an element for some $1 \leq u \leq k$ and $1 \leq v \leq m_u$ and let $I = \{1, \dots, m\}$. We take a division quaternion algebra (x, y) over a field extension K/F. Then, choose ϕ_i for all $i \in I$ such that

$$\phi[e(i_u)] = \phi[e(i_{u,q})] = \langle x, y, xy\rangle \perp h,\ \phi[e(i_{u,v})] = \langle 1, z, xz, yz, xyz\rangle \perp h \tag{52}$$

for all $1 \leq q \neq v \leq m_u$,

$$\phi[e(i_p)] = \phi[e(i_{p,q})] = \begin{cases} \langle x, y, xy\rangle \perp h & \text{if } e(i_u) \text{ appears in } b(i_p), \\ \langle 1\rangle \perp h & \text{otherwise,} \end{cases}$$

for all $i_p \in J_2'$ and all q with $e(i_p, i_{p,q}) \in B_2$, and $\phi_i = \langle 1\rangle \perp h$ for the remaining $i \in I$ over $K((z))$, where h denotes a hyperbolic form depending on the dimension of each ϕ_i. Then, we have

$$\phi[e(i_u, i_{u,v})] = \langle\langle x, y, z\rangle\rangle,\ \phi[e(i_u, i_{u,q})] = \langle\langle x, y, 1\rangle\rangle \tag{53}$$

for all $1 \leq q \neq v \leq m_u$,

$$\phi[e(i_p, i_{p,q})] = \phi[b(i_p)] = \langle\langle x, y, 1\rangle\rangle \tag{54}$$

for all $p \in J_2'$ and all q with $e(i_p, i_{p,q}) \in B_2$ such that $e(i_u)$ appears in $b(i_p)$, and $\phi[b] = 0$ for all remaining $b \in B_2 \cup B_3$ in the Witt ring of $K((z))$. Therefore, we obtain $\partial_z(\alpha(\phi)) = (x, y) \neq 0$. Hence, $\alpha(\phi)$ ramifies.

Case 2: $\exists\, e(i_p, i_{p,q}) \in B_2'$ with $n_{i_p} + n_{i_{p,q}} \geq 3$ such that $i_p \in J_2'$. Let $e(i_u, i_{u,v}) \in B_2'$ be such an element as in the previous case. Observe that by construction of B_3 there exists

$$k_1 \in I \backslash \{i_p, j_r \mid i_p \in J_2', 1 \leq r \leq s\} \tag{55}$$

such that $e(k_1)$ appears in $b'(i_u)$. We first choose $\phi[e(i_{u,v})]$ as in (52) and $\phi[e(k_1)] = \langle x, y, xy\rangle \perp h$. Then, we choose ϕ_i for $i \in I \backslash \{i_{u,v}, k_1\}$ such that

$$\phi[e(i)] = \begin{cases} \langle x, y, xy\rangle \perp h & \text{if } e(k_1) \text{ appears in } b(i), \\ \langle 1\rangle \perp h & \text{otherwise} \end{cases}$$

for all $i \in \{i_p, j_r \,|\, i_p \in J_2', 1 \leq r \leq s\}$,

$$\phi[e(i_{p,q})] = \begin{cases} \langle x, y, xy\rangle \perp h & \text{if } i_p = k_1 \text{ or } e(k_1) \text{ appears in } b(i_p), \\ \langle 1 \rangle \perp h & \text{otherwise} \end{cases}$$

for all q such that $e(i_p, i_{p,q}) \in B_2$, and $\phi_i = \langle 1\rangle \perp h$ for the remaining $i \in I \backslash \{i_{u,v}, k_1\}$ over $K((z))$. Therefore, we obtain (53),

$$\phi[b(i)] = \phi[e(i_p, i_{p,q})] = \langle\langle x, y, 1\rangle\rangle \tag{56}$$

for all $i \in \{i_p, j_r \,|\, i_p \in J_2', 1 \leq r \leq s\}$ such that $e(k_1)$ appears in $b(i)$ and for all $e(i_p, i_{p,q}) \in B_2$ such that $i_p = k_1$ or $e(k_1)$ appears in $b(i_p)$, and $\phi[b] = 0$ for all remaining $b \in B_2 \cup B_3$ in the Witt ring of $K((z))$. Hence, $\partial_z(\alpha(\phi)) = (x, y) \neq 0$, thus $\alpha(\phi)$ ramifies. □

We present the second main result on the group of unramified degree 3 invariants for type B.

Theorem 6.5. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i+1})/\boldsymbol{\mu}$ defined over an algebraically closed field F, $m, n_i \geq 1$, where $\boldsymbol{\mu}$ is a central subgroup. Then, every unramified degree 3 invariant of G is trivial, i.e., $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.*

Proof. Let $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{\Gamma}_{2n_i+1})/\boldsymbol{\mu}$. Since the classifying space BG is stably birational to the classifying space BG_{red}, by (1) we have $\mathrm{Inv}^3_{\mathrm{nr}}(G) = \mathrm{Inv}^3_{\mathrm{nr}}(G_{\mathrm{red}})$. We shall show that $\mathrm{Inv}^3_{\mathrm{nr}}(G_{\mathrm{red}}) = 0$. Let $G' = (\mathbf{Spin}_3)^m/\boldsymbol{\mu}$ and $G'_{\mathrm{red}} = (\mathbf{\Gamma}_3)^m/\boldsymbol{\mu}$. Then, the standard embeddings $\mathbf{Spin}_3 \to \mathbf{Spin}_{2n_i+1}$ and $\mathbf{\Gamma}_3 \to \mathbf{\Gamma}_{2n_i+1}$ induce morphisms $G' \to G$ and $G'_{\mathrm{red}} \to G_{\mathrm{red}}$, thus we have

$$\begin{tikzcd} \mathrm{Inv}^3(G) \arrow[r] & \mathrm{Inv}^3(G') \\ \mathrm{Inv}^3(G_{\mathrm{red}}) \arrow[r] \arrow[u, hook] & \mathrm{Inv}^3(G'_{\mathrm{red}}) \arrow[u, hook] \end{tikzcd} \tag{57}$$

By (50) in Proposition 6.3 and Lemma 6.4 we may assume that the bottom map in (57) is an isomorphism. By [14, Lemma 4.3] and the exceptional isomorphism $A_1 = B_1$, we have $\mathrm{Inv}^3_{\mathrm{nr}}(G'_{\mathrm{red}}) = 0$, thus every invariant of G_{red} is ramified. □

6.2. **Type C.**

Lemma 6.6. *Let $G = (\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$, $m, n_i \geq 1$, where $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^* = (\mathbb{Z}/2\mathbb{Z})^m$ whose quotient is the character group $\boldsymbol{\mu}^*$. Set $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{GSp}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{GSp}_{2n_i}$ denotes the group of symplectic similitudes. Then, for any field extension K/F the first Galois cohomology set $H^1(K, G_{\mathrm{red}})$ is bijective to the set of m-tuples $\big((A_1, \sigma_1), \dots, (A_m, \sigma_m)\big)$ of pairs of central simple K-algebra A_i of degree $2n_i$ with symplectic involution σ_i such that for all $r = (r_i) \in R$*

$$r_1 A_1 + \cdots + r_m A_m = 0 \textit{ in } \mathrm{Br}(K),$$

where $\mathrm{Br}(K)$ denotes the Brauer group of K.

Proof. Let $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{GSp}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{GSp}_{2n_i}$ denotes the group of symplectic similitudes. Consider the exact sequence

$$1 \to (\mathbb{G}_m)^m/\boldsymbol{\mu} \to G_{\mathrm{red}} \to \prod_{i=1}^{m} \mathbf{PGSp}_{2n_i} \to 1.$$

Then, by the same argument as in the proof of Lemma 6.1 the set $H^1(F, G_{\mathrm{red}})$ is bijective to the kernel of following map

$$H^1(F, \prod_{i=1}^{m} \mathbf{PGSp}_{2n_i}) \to \mathrm{Br}_2(F)^m \xrightarrow{\tau} H^2(F, (\boldsymbol{\mu}_2)^m/\boldsymbol{\mu}),$$

where the first map sends an m-tuple $\big((A_1, \sigma_1), \dots, (A_m, \sigma_m)\big)$ of simple algebra A_i of degree $2n_i$ with symplectic involution σ_i to the m-tuple $(A_1, \dots, A_m)$ and the map τ is induced by the natural surjection $(\boldsymbol{\mu}_2)^m \to (\boldsymbol{\mu}_2)^m/\boldsymbol{\mu}$. Since $(A_1, \dots, A_m) \in \mathrm{Ker}(\tau)$ if and only if it is contained in the kernel of the map in (48) for all $r \in R$, thus we have

$$H^1(F, G_{\mathrm{red}}) \simeq \{\big((A_1, \sigma_1), \dots, (A_m, \sigma_m)\big) \mid \deg A_i = 2n_i, \sum_{i=1}^{m} r_i A_i = 0\} \tag{58}$$

for all $r = (r_i) \in R$. □

Let (A, σ) be a pair of central simple F-algebra A of degree $2n$ with involution σ of the first kind. The trace form $T_\sigma : A \to F$ is given by $T_\sigma(a) = \mathrm{Trd}(\sigma(a)a)$, where Trd denotes the reduced trace. We denote by T_σ^+ the restriction of T_σ to $\mathrm{Sym}(A, \sigma)$. Set

$$\phi[r] := \perp_{i=1}^{m} r_i \phi_i, \text{ where } \phi_i = \begin{cases} T_{\sigma_i} & \text{if } n_i \equiv 1 \mod 2, \\ T_{\sigma_i}^+ & \text{if } n_i \equiv 2 \mod 4 \end{cases} \tag{59}$$

for all $r = (r_i) \in R \cap (\bigoplus_{4\nmid n_i} (\mathbb{Z}/2\mathbb{Z}) e_i)$. For all $i \in I$ such that $4|n_i$, we simply write Δ for the Garibaldi-Parimala-Tignol invariant $\Delta(A_i, \sigma_i)$ defined in [11, Theorem A]. Then, this degree 3 invariant induces the following invariants of G_{red}

$$\Delta_i : H^1(K, G_{\mathrm{red}}) \to H^1(K, \mathbf{PGSp}_{2n_i}) \xrightarrow{\Delta} H^3(K), \tag{60}$$

where the first map in (60) is the projection and K/F is a field extension. We show that every invariant of semisimple group of type C is generated by the Arason invariants associated to $\phi[r]$ and the Garibaldi-Parimala-Tignol invariants Δ_i.

Proposition 6.7. *Let $G = (\prod_{i=1}^{m} \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ defined over an algebraically closed field F, where $m, n_i \geq 1$, $\boldsymbol{\mu}$ is a central subgroup. Let R be the subgroup of $(\boldsymbol{\mu}_2^m)^*$ whose quotient is the character group $\boldsymbol{\mu}^*$. Set $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{GSp}_{2n_i})/\boldsymbol{\mu}$. Then, every normalized invariant in $\mathrm{Inv}^3(G_{\mathrm{red}})$ is of the form*

$$\sum_{r \in R'} \mathbf{e}_3(\phi[r]) + \sum_{i \in I'} \Delta_i \tag{61}$$

for some $R' \subseteq R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i)$ *and some subset* $I' \subseteq \{i \in I \,|\, 4|n_i\}$*, where* $\phi[r]$ *denotes the quadratic form defined in (59) and* $\mathbf{e}_3 : I^3(K) \to H^3(K)$ *denotes the Arason invariant over a field extension* K/F*. Moreover, we have*

$$\mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}} \simeq \frac{\bigoplus_{4\,|n_i}(\mathbb{Z}/2\mathbb{Z})e_i \bigoplus \big(R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i)\big)}{\langle e_i,\, e_j + e_k \in R \,|\, e_j, e_k \notin R,\, n_j \equiv n_k \equiv 1 \mod 2\rangle}. \tag{62}$$

Proof. Since F is algebraically closed, we get $\mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}} = \mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{ind}}$. Let i be an integer such that $n_i \equiv 0 \mod 4$. If $e_i \in R$, then, as every symplectic involution on a split algebra is hyperbolic, by Lemma 6.6 and [11, Theorem A] the invariant Δ_i defined in (60) vanishes. Now assume that $e_i \notin R$. Let $Q = (x, y)$ be a division quaternion algebra over a field extension K/F and let $b = \langle 1, z\rangle \perp h$ be a symmetric bilinear form on E^{n_i}, where h denotes a hyperbolic form and $E = K((z))$. Consider the linear system as in (58) with the coefficients given by a basis of R. As $e_i \notin R$, it follows by the rank theorem (or Rouché-Capelli theorem) that there exists a G_{red}-torsor $\eta = \big((A_1, \sigma_1), \dots, (A_m, \sigma_m)\big)$ over E such that

$$(A_i, \sigma_i) = (M_{n_i}(Q), \sigma_b \otimes \gamma) \text{ and } (A_j, \sigma_j) = (M_{2n_j}(E), \sigma_\omega) \text{ or } (M_{n_j}(Q), t \otimes \gamma) \tag{63}$$

for all $1 \le j \ne i \le m$, where γ denotes the canonical involution on Q, σ_b denotes the adjoint involution on $\mathrm{End}(E^{n_i}) = M_{n_i}(E)$ with respect to b, σ_ω denotes the adjoint involution with respect to the standard symplectic bilinear form ω, and t denotes the transpose involution on $M_{n_j}(E)$. Then, by [11, Example 3.1] we have

$$\Delta_i(\eta) = (Q) \cup (z), \tag{64}$$

thus, $\partial_z(\alpha(\eta)) = (x, y) \ne 0$. Therefore, we have a nontrivial invariant Δ_i of order 2 for any i such that $n_i \equiv 0 \mod 4$ and $e_i \notin R$.

Let $r \in R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i)$. Since each quadratic form ϕ_i in (59) has even dimension and trivial discriminant, we obtain $\phi[r] \in I^2(K)$ for each r. By [20, Theorem 1] the Hasse invariant of ϕ_i in (59) coincides with the class of A_i in $\mathrm{Br}(K)$, thus by the relation in (58), we have $\phi[r] \in I^3(K)$ for each $r \in R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i)$. Therefore, the Arason invariant induces a normalized invariant $\mathbf{e}_3(\phi[r])$ of order dividing 2 that sends an m-tuple in (58) to $\mathbf{e}_3(\phi[r]) \in H^3(K)$.

Let $r \in R_1'' + R_2''$, where $R_1'' = \langle e_i \in R \,|\, n_i \not\equiv 0 \mod 4\rangle$ and R_2'' denotes the subgroup of R defined in Proposition 4.2. For any $e_i \in R_1''$ and any $e_j + e_k \in R_2''$, we have

$$\phi_i = T_{\sigma_i} = h \text{ and } \phi_j \perp \phi_k = T_{\sigma_j} \perp T_{\sigma_k} = \langle\langle a, b, 1\rangle\rangle \perp h', \tag{65}$$

where $A_j = A_k = (a, b)$ in $\mathrm{Br}(K)$, h and h' denote hyperbolic forms, thus both invariants $\mathbf{e}_3(\phi[e_i])$ and $\mathbf{e}_3(\phi[e_j + e_k])$ vanish. Therefore, the invariant $\mathbf{e}_3(\phi[r])$ vanishes.

To complete the proof, by Theorem 5.5 it suffices to show that the invariant $\mathbf{e}_3(\phi[r])$ is nontrivial for any $r \in R \cap (\bigoplus_{4\nmid n_i}(\mathbb{Z}/2\mathbb{Z})e_i) \backslash (R_1'' + R_2'')$. Let $G'_{\mathrm{red}} = (\mathbf{GSp}_2)^m/\boldsymbol{\mu}$. Then, the rest of the proof of Proposition 6.3 still works if we replace the exceptional isomorphism $A_1 = B_1$, the standard embedding $\mathbf{\Gamma}_3 \to \mathbf{\Gamma}_{2n_i+1}$, and Lemma 6.4 in the proof of Proposition 6.3 by the exceptional isomorphism $A_1 = C_1$, the standard embedding $\mathbf{GSp}_2 \to \mathbf{GSp}_{2n_i}$, and Lemma 6.9, respectively. □

Remark 6.8. If $m = 2$, $n_1 \equiv n_2 \equiv 0 \mod 2$, and $\boldsymbol{\mu} \subseteq \boldsymbol{\mu}_2^2$ is the diagonal subgroup, then the invariant in Proposition 6.7 coincides with the invariant defined in [3].

We present the following analogue of Lemma 6.4, which plays the same role for the triviality of unramified invariants as Lemma 6.4 plays for the groups of type B.

Lemma 6.9. *Let $G = (\prod_{i=1}^{m} \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ defined over an algebraically closed field F, where $m, n_i \geq 1$, $\boldsymbol{\mu}$ is a central subgroup. Set $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{GSp}_{2n_i})/\boldsymbol{\mu}$. Then, every normalized invariant in $\operatorname{Inv}^3(G_{\mathrm{red}})$ is ramified if either n_i is divisible by 4 for some i with $e_i \notin R_1$ or $n_j n_k \not\equiv 1 \mod 2$ for some j and k such that $e_j + e_k \in R \cap (\bigoplus_{4 \nmid n_i} (\mathbb{Z}/2\mathbb{Z}) e_i)$.*

Proof. Let α be a normalized invariant in $\operatorname{Inv}^3(G_{\mathrm{red}})$ be written as in (61) for some subspace $R' \subseteq R \cap (\bigoplus_{4 \nmid n_i} (\mathbb{Z}/2\mathbb{Z}) e_i)$ and subset $I' \subseteq \{i \in I \mid n_i \equiv 0 \mod 4, e_i \notin R\}$.

Assume that there exist $i \in I'$. Let $\eta = \big((A_1, \sigma_1), \dots, (A_m, \sigma_m)\big)$ be a G_{red}-torsor as in the proof of Proposition 6.7. Then, by (63), [11, Example 3.1], and [11, Theorem A] we have

$$\Delta_j(\eta) = 0$$

for all $j \neq i$ such that $n_j \equiv 0 \mod 4$. Since

$$\phi_j = \begin{cases} h & \text{if } (A_j, \sigma_j) = (M_{2n_j}(E), \sigma_\omega), \\ \langle\langle x, y \rangle\rangle \perp h & \text{if } (A_j, \sigma_j) = (M_{n_j}(Q), t \otimes \gamma), \end{cases}$$

where h denotes a hyperbolic form and the pairs of the form $(M_{n_j}(Q), t \otimes \gamma)$ appear an even number of times in the relation of (58) for any $r \in R'$, we have $\mathbf{e}_3(\phi[r]) = 0$ for any $r \in R'$. Therefore, by (64) we have $\partial_z(\alpha(\eta)) = (x, y) \neq 0$, thus the invariant α ramifies.

We may assume that $n_i \not\equiv 0 \mod 4$ for all $1 \leq i \leq m$, thus

$$\alpha(\eta) = \mathbf{e}_3(\phi[r_2]) + \mathbf{e}_3(\phi[r_3])$$

for some nonzero $r_2 \in R_2$ and some $r_3 \in R_3$, where R_1 and R_2 denote the subspaces of R in (12), R_3 is a complementary subspace of $R_1 + R_2$ in R, and η is a G_{red}-torsor. Then, we choose bases $B_2 = \{e(i_p, i_{p,q})\}$ of R_2 with $n_{i_{p,q}} \geq n_{i_p}$ and B_3 of a complementary subspace of R_2 as in Lemma 6.4 so that the invariant α is written as in (51). We show that the invariant $\alpha(\eta)$ ramifies following the proof of Lemma 6.4.

Case 1: $\exists\ e(i_p, i_{p,q}) \in B_2'$ with $n_{i_p} n_{i_{p,q}} \not\equiv 1 \mod 2$ such that $i_p \notin J_2'$. Let $e(i_u, i_{u,v}) \in B_2'$ be such an element for some $1 \leq u \leq k$ and $1 \leq v \leq m_u$. Let $Q = (x, y)$ be a division quaternion algebra over K/F and let $Q_1 = (x, z)$ and $Q_2 = (x, yz)$ be quaternions over E. We denote by γ, γ_1, γ_2 the canonical involutions on Q, Q_1, Q_2, respectively. For the sake of simplicity, we shall write the symbol d for the corresponding degree of the matrix algebras in the rest of the proof. Now we choose $\eta = \big((A_i, \sigma_i)\big)$ for $i \in I$ such that

$$(A_i, \sigma_i) = (M_d(Q), t \otimes \gamma),\ (A_{i_{u,v}}, \sigma_{i_{u,v}}) = (M_d(Q_1 \otimes Q_2), t \otimes \gamma_1' \otimes \gamma_2) \tag{66}$$

for $i = i_u, i_{u,q}$ and all $1 \leq q \neq v \leq m_u$, where t denotes the transpose involution on a matrix algebra and γ_1' denotes an orthogonal involution on Q_1 given by the composition of γ_1 and the inner automorphism induced by one of the generators of pure quaternions in Q_1,

$$(A_{i_p}, \sigma_{i_p}), (A_{i_{p,q}}, \sigma_{i_{p,q}}) = \begin{cases} (M_d(Q), t\otimes\gamma) & \text{if } e(i_u) \text{ appears in } b(i_p), \\ (M_d(E), \sigma_\omega) & \text{otherwise,} \end{cases} \tag{67}$$

for all $i_p \in J_2'$ and all q with $e(i_p, i_{p,q}) \in B_2$, and

$$(A_i, \sigma_i) = (M_d(E), \sigma_\omega) \tag{68}$$

for the remaining $i \in I$. Then, we have

$$\phi[e(i_u)] = \phi[e(i_{u,q})] = \langle\langle x, y\rangle\rangle \perp h,\ \phi[e(i_{u,v})] = \langle z, xz, yz, xyz\rangle \perp h$$

for all $1 \leq q \neq v \leq m_u$, thus we obtain (53), (54), and $\phi[b] = 0$ for all remaining $b \in B_2 \cup B_3$ in the Witt ring of E. Hence, $\partial_z(\alpha(\eta)) = (x, y) \neq 0$, i.e., α ramifies.

Case 2: $\exists\ e(i_p, i_{p,q}) \in B_2'$ with $n_{i_p} n_{i_{p,q}} \not\equiv 1 \mod 2$ such that $i_p \in J_2'$. Let $e(i_u, i_{u,v}) \in B_2'$ be such an element. We choose k_1 as in (55) and then choose (A_{k_1}, σ_{k_1}) and $(A_{i_{u,v}}, \sigma_{i_{u,v}})$ as in (66). Then, we choose (A_i, σ_i) for $i \in I\backslash\{i_{u,v}, k_1\}$ such that

$$(A_i, \sigma_i) = \begin{cases} (M_d(Q), t\otimes\gamma) & \text{if } e(k_1) \text{ appears in } b(i), \\ (M_d(E), \sigma_\omega) & \text{otherwise} \end{cases} \tag{69}$$

for all $i \in \{i_p, j_r \mid i_p \in J_2', 1 \leq r \leq s\}$,

$$(A_{i_{pq}}, \sigma_{i_{p,q}}) = \begin{cases} (M_d(Q), t\otimes\gamma) & \text{if } i_p = k_1 \text{ or } e(k_1) \text{ appears in } b(i_p), \\ (M_d(E), \sigma_\omega) & \text{otherwise} \end{cases} \tag{70}$$

for all q such that $e(i_p, i_{p,q}) \in B_2$, and

$$(A_i, \sigma_i) = (M_d(E), \sigma_\omega) \tag{71}$$

for the remaining $i \in I\backslash\{i_{u,v}, k_1\}$. Therefore, we obtain (53), (56), and $\phi[b] = 0$ for all remaining $b \in B_2 \cup B_3$ in the Witt ring of E. Therefore, $\partial_z(\alpha(\eta)) = (x, y) \neq 0$, thus α ramifies. □

We show that the same result in Theorem 6.5 holds for the groups of type C.

Theorem 6.10. *Let $G = (\prod_{i=1}^m \mathbf{Sp}_{2n_i})/\boldsymbol{\mu}$ defined over an algebraically closed field F, $m, n_i \geq 1$, where $\boldsymbol{\mu}$ is a central subgroup. Then, every unramified degree 3 invariant of G is trivial, i.e., $\operatorname{Inv}^3_{\mathrm{nr}}(G) = 0$.*

Proof. Let $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{GSp}_{2n_i})/\boldsymbol{\mu}$, $G'_{\mathrm{red}} = (\mathbf{GSp}_2)^m/\boldsymbol{\mu}$, and $G' = (\mathbf{Sp}_2)^m/\boldsymbol{\mu}$. Then, the proof of Theorem 6.5 still works if we replace Proposition 6.3, Lemma 6.4, and the exceptional isomorphism $A_1 = B_1$ in the proof by Proposition 6.7, Lemma 6.9, and the exceptional isomorphism $A_1 = C_1$, respectively. □

6.3. **Type D.**

Lemma 6.11. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$, $m, \geq 1$, $n_i \geq 3$, where $\boldsymbol{\mu}$ is a central subgroup.Let R be the subgroup of the character group Z defined in (17) such that $\boldsymbol{\mu}^* = Z/R$. Set $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{\Omega}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{\Omega}_{2n_i}$ denotes the extended Clifford group. Then, for any field extension K/F the first Galois cohomology set $H^1(K, G_{\mathrm{red}})$ is bijective to the set of m-tuples $\big((A_1, \sigma_1, f_1), \dots, (A_m, \sigma_m, f_m)\big)$ of triples consisting of a central simple K-algebra A_i of degree $2n_i$ with orthogonal involution σ_i of trivial discriminant and a K-algebra isomorphism $f_i : Z(C(A_i, \sigma_i)) \simeq K \times K$, where $Z\big(C(A_i, \sigma_i)\big)$ denotes the center of the Clifford algebra $C(A_i, \sigma_i)$, satisfying*

$$B_1 + \cdots + B_m = 0 \text{ in } \mathrm{Br}(K)$$

for all $\sum_{i=1}^{m} r_i' \in R$ with

$$r_i' = \begin{cases} r_i e_i & \text{if } n_i \text{ odd,} \\ r_{i,1}e_{i,1} + r_{i,2}e_{i,2} & \text{if } n_i \text{ even,} \end{cases}$$

where

$$B_i := \begin{cases} r_i C_{i,1} \text{ or } r_i C_{i,2} & \text{if } n_i \text{ odd,} \\ r_{i,1}C_{i,1} + r_{i,2}C_{i,2} \text{ or } r_{i,1}C_{i,2} + r_{i,2}C_{i,1} & \text{if } n_i \text{ even} \end{cases}$$

depending on the choice of two isomorphisms f_i for each triple (A_i, σ_i, f_i), $C_{i,1}$ and $C_{i,2}$ denote simple K-algebras such that $C(A_i, \sigma_i) = C_{i,1} \times C_{i,2}$, and $\mathrm{Br}(K)$ denotes the Brauer group of K.

Proof. Let $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{\Omega}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{\Omega}_{2n_i}$ denotes the extended Clifford group ([12, §13]). Consider the exact sequence

$$1 \to (\mathbb{G}_m)^{2m}/\boldsymbol{\mu} \to G_{\mathrm{red}} \to \prod_{i=1}^{m} \mathbf{PGO}_{2n_i}^{+} \to 1,$$

where $\mathbf{PGO}_{2n_i}^{+}$ denotes the projective orthogonal group. Applying the same argument as in the proof of Lemma 6.1 we see that the set $H^1(K, G_{\mathrm{red}})$ is bijective to the kernel of following map

$$H^1(K, \prod_{i=1}^{m} \mathbf{PGO}_{2n_i}^{+}) \xrightarrow{\beta} \mathrm{Br}\left(Z(\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})\right) \xrightarrow{\tau} H^2(K, Z(\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}),$$

where the map β sends an m-tuple $\big((A_i, \sigma_i, f_i)\big)$ of triples consisting of a central simple K-algebra A_i of degree $2n_i$ with orthogonal involution σ_i of trivial discriminant and a K-algebra isomorphism $f_i : Z(C(A_i, \sigma_i)) \simeq K \times K$ to the m-tuple $(B_1', \dots, B_m')$ with

$$B_i' := \begin{cases} C_{i,1} \text{ or } C_{i,2} & \text{if } n_i \text{ odd,} \\ (C_{i,1}, C_{i,2}) \text{ or } (C_{i,2}, C_{i,1}) & \text{if } n_i \text{ even,} \end{cases}$$

depending on the choice of two isomorphisms f_i for each triple (A_i, σ_i, f_i) (i.e., For odd (resp. even) n_i, the image of (A_i, σ_i, f_i) under β is $C_{i,1}$ (resp. $(C_{i,1}, C_{i,2})$) if and only if the image of (A_i, σ_i, f_i') for another isomorphism $f_i' : Z\big(C(A_i, \sigma_i)\big) \simeq K \times K$

under β is $C_{i,2}$ (resp. $(C_{i,2}, C_{i,1})$)) and the map τ is induced by the natural surjection $Z(\prod_{i=1}^m \mathbf{Spin}_{2n_i}) \to Z(\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$. As $(B'_1, \ldots, B'_m) \in \mathrm{Ker}(\tau)$ if and only if it is contained in the kernel of the composition

$$H^2(K, Z(\prod_{i=1}^m \mathbf{Spin}_{2n_i}) \xrightarrow{\tau} H^2(K, Z(\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}) \xrightarrow{r_*} H^2(K, \mathbb{G}_m)$$

for all $r \in R = (Z(\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu})^*$, we obtain

$$H^1(K, G_{\mathrm{red}}) \simeq \{\big((A_i, \sigma_i, f_i)\big) \mid \sum_{i=1}^m B_i = 0 \text{ in } \mathrm{Br}(K)\} \tag{72}$$

for all $\sum_{i=1}^m r'_i \in R$. □

Recall from Theorem 5.6 the following subsets

$I_1 = \{i \mid Z_i = R_{1,i} \text{ or } R'_{1,i}, n_i \neq 3\}$ and

$I_2 = \{i \mid R'_{1,i} = 0, 4|n_i\} \cup \{i \mid R'_{1,i} = (\mathbb{Z}/2\mathbb{Z})e_{i,1} \text{ or } (\mathbb{Z}/2\mathbb{Z})e_{i,2}, n_i \geq 6, 4|n_i\} =: I_{21} \cup I_{22}$.

Let $i \in I_1$. Then, from Lemma 6.11, we see that both K-algebras A_i and $C(A_i, \sigma_i)$ split, thus we have $(A_i, \sigma_i, f_i) \simeq (M_{2n_i}(K), \sigma_{\psi_i})$ for some adjoint involution σ_{ψ_i} with respect to a quadratic form ψ_i such that $\psi_i \in I^3(K)$. Hence, the Arason invariant $\mathbf{e_3}$ induces the following invariant

$$\mathbf{e}_{3,i} : H^1(K, G_{\mathrm{red}}) \to H^3(K) \tag{73}$$

given by $\mathbf{e}_{3,i}\big((A_1, \sigma_1, f_1), \ldots, (A_m, \sigma_m, f_m)\big) = \mathbf{e_3}(\psi_i)$. This invariant is obviously nontrivial.

Now let $i \in I_2$. Then, the invariant Δ' of $\mathbf{PGO}^+_{2n}$ ([19, Theorem 4.7]) gives the following invariant of G_{red}

(74)

$$\Delta'_i : \begin{cases} H^1(K, G_{\mathrm{red}}) \to H^1(K, \mathbf{PGO}^+_{2n_i}) \xrightarrow{\Delta'} H^3(K) & \text{if } i \in I_{21}, \\ H^1(K, G_{\mathrm{red}}) \to H^1(K, \mathbf{HSpin}_{2n_i}) \to H^1(K, \mathbf{PGO}^+_{2n_i}) \xrightarrow{\Delta'} H^3(K) & \text{if } i \in I_{22}, \end{cases}$$

where $\mathbf{HSpin}_{2n_i}$ denotes the half-spin group and the first map in (74) is the projection for each case.

We shall need the following analogue of [11, Example 3.1].

Lemma 6.12. *Let Q be a quaternion algebra over F and let $(A, \sigma, f) \in H^1(F, \mathbf{PGO}^+_{2n})$ such that $n \equiv 0 \mod 2$ and $(A, \sigma) = (M_n(F) \otimes Q, \sigma_1 \otimes \sigma_2)$ for some orthogonal involutions σ_1 and σ_2 on $M_n(F)$ and Q, respectively. Then, we have*

$$\Delta'(A, \sigma, f) = Q \cup (\operatorname{disc} \sigma_1).$$

Proof. Let t be the transpose involution on $M_n(F)$. Since $\sigma_1 = \mathrm{Int}(x) \circ t$ for some t-symmetric invertible element x, where $\mathrm{Int}(x)$ denotes the inner automorphism induced by x, we have

$$\mathrm{disc}(\sigma_1) = \mathrm{Nrd}_{M_n(F)}(x) = \sqrt{\mathrm{Nrd}_A(x \otimes 1)}$$

and $\sigma = \mathrm{Int}(x \otimes 1) \circ (t \otimes \sigma_2)$, where Nrd denotes the reduced norm. As $x \otimes 1$ is a σ-symmetric invertible element, the result follows from [19, §4b]. $\square$

Proposition 6.13. *Let $G = (\prod_{i=1}^{m} \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ defined over an algebraically closed field F, where $m \geq 1$, $n_i \geq 3$, $\boldsymbol{\mu}$ is a central subgroup. Set $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{\Omega}_{2n_i})/\boldsymbol{\mu}$, where $\mathbf{\Omega}_{2n_i}$ is the extended Clifford group. Then, every normalized invariant in $\mathrm{Inv}^3(G_{\mathrm{red}})$ is of the form*

$$\sum_{i\in I_1'} \mathbf{e}_{3,i} + \sum_{i\in I_2'} \Delta_i' + \sum_{r\in R''} \mathbf{e}_3(\phi[r]) \tag{75}$$

for some subsets $I_1' \subseteq I_1$, $I_2' \subseteq I_2$, and $R'' \subseteq R'$, where R' denotes the group as defined in Theorem 5.6, $\phi[r]$ is the quadratic form defined in (59) and $\mathbf{e}_3 : I^3(K) \to H^3(K)$ denotes the Arason invariant for a field extension K/F. Moreover, we have

$$\mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}} \simeq \frac{\bigoplus_{i\in I_1\cup I_2} (\mathbb{Z}/2\mathbb{Z})\bar{e}_i \bigoplus R'}{\langle \bar{e}_i,\ \bar{e}_j + \bar{e}_k \in R' \mid \bar{e}_j, \bar{e}_k \notin R',\ n_j \equiv n_k \equiv 1 \mod 2\rangle}.$$

Proof. Since F is algebraically closed, we obtain $\mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{norm}} = \mathrm{Inv}^3(G_{\mathrm{red}})_{\mathrm{ind}}$. We first show that the invariant Δ_j' is nontrivial for all $j \in I_2$. Choose a field extension K/F containing variables $x_{i,1}$, $x_{i,2}$, x_i, y_i, division quaternion K-algebras

$$Q_{i,1} = (x_{i,1}, y_i),\ Q_{i,2} = (x_{i,2}, y_i)$$

for all $i \in I$ such that $n_i \equiv 0 \mod 2$, and cyclic division K-algebras

$$P_i = (x_i, y_i)_4$$

of exponent 4 for all $i \in I$ such that $n_i \equiv 1 \mod 2$. Let

$$Q_i = \begin{cases} (x_{i,1}x_{i,2},\ y_i) & \text{if } n_i \text{ even,} \\ (x_i,\ y_i) & \text{if } n_i \text{ odd,} \end{cases} \quad \text{so that } Q_i = \begin{cases} Q_{i,1} + Q_{i,2} & \text{if } n_i \text{ even,} \\ 2P_i & \text{if } n_i \text{ odd} \end{cases}$$

in $\mathrm{Br}(K)$. For $r \in R$, let

$$D_{1,r} = \bigotimes_{2\mid n_i} (Q_{i,1}^{r_{i,1}} \otimes Q_{i,2}^{r_{i,2}}),\ D_{2,r} = \bigotimes_{2\nmid n_i} P_i^{r_i},\ \text{and } D_r = D_{1,r} \otimes D_{2,r}.$$

Let L be the function field of the product $\prod_{r\in R} \mathrm{SB}(D_r)$ of Severi-Brauer varieties $\mathrm{SB}(D_r)$ of D_r over K. For all i such that $n_i \equiv 1 \mod 2$, consider the exterior square $\lambda^2 P_i$ of P_i with its canonical involution ρ_i [12, §10]. By the exceptional isomorphism $A_3 = D_3$ ([12, 15.32]) we have

$$C(\lambda^2 P_i, \rho_i) = P_i \times P_i^{\mathrm{op}}, \tag{76}$$

where P_i^{op} denotes the opposite algebra of P_i. Let χ_i be a skew-hermitian form over Q_i such that $(M_3(Q_i), \sigma_{\chi_i}) = (\lambda^2 P_i, \rho_i)$, where σ_{χ_i} is the adjoint involution with respect to χ_i. Let $\psi_i = \chi_i \perp h$ be a skew-hermitian form over Q_i of rank n_i, where h denotes a hyperbolic form (if $n_i = 3$, then $\psi_i = \chi_i$). We denote by σ_{ψ_i} the adjoint involution on $M_{n_i}(Q_i)$ with respect to ψ_i. Let

$$(A_i, \sigma_i) = \begin{cases} (M_{n_i}(L) \otimes Q_i, \sigma_{i,1} \otimes \sigma_{i,2}) & \text{if } n_i \text{ even,} \\ (M_{n_i}(Q_i), \sigma_{\psi_i}) & \text{if } n_i \text{ odd} \end{cases}$$

for some orthogonal involutions $\sigma_{i,1}$ on $M_{n_i}(L)$ and $\sigma_{i,2}$ on Q_i such that $\mathrm{disc}(\sigma_{i,1}) = x_{i,1}$ and $\mathrm{disc}(\sigma_{i,2}) = y_i$. Then, by [8, Theorem 1.1] and [7, Corollary 3] together with (76) we obtain

$$C(A_i, \sigma_i) = \begin{cases} M_{2^{n_i-2}}(Q_{i,1}) \times M_{2^{n_i-2}}(Q_{i,2}) & \text{if } n_i \text{ even,} \\ M_{2^{n_i-3}}(P_i) \times M_{2^{n_i-3}}(P_i)^{\mathrm{op}} & \text{if } n_i \text{ odd,} \end{cases}$$

thus by a theorem of Amitsur we have a $G_{\mathrm{red}}(L)$-torsor $\eta = \big((A_i, \sigma_i, f_i)\big)$. Finally, by Lemma 6.12 we get $\Delta'_j(\eta) = (x_{j,1}, x_{j,2}, y_j) \neq 0$.

Now, let $r = (\bar{r}_1, \ldots, \bar{r}_m) \in R'$. Then, from Lemma 6.11 we have

$$B_i = A_i = \begin{cases} 2\bar{r}_i C_{i,1} = 2\bar{r}_i C_{i,2} & \text{if } n_i \text{ odd,} \\ \bar{r}_i C_{i,1} + \bar{r}_i C_{i,2} & \text{if } n_i \text{ even,} \end{cases}$$

in $\mathrm{Br}(K)$, thus the relation in (72) is equivalent to

$$\bar{r}_1 A_1 + \cdots + \bar{r}_m A_m = 0. \tag{77}$$

As each quadratic form ϕ_i in (59) has even dimension and trivial discriminant, we have $\phi[r] \in I^2(K)$ for each $r \in R'$. By [20, Theorem 1] the Hasse invariant of ϕ_i in (59) coincides with the class of A_i in $\mathrm{Br}(K)$, thus by the relation in (77), we have $\phi[r] \in I^3(K)$ for each $r \in R'$. Therefore, the Arason invariant induces a normalized invariant $\mathbf{e}_3(\phi[r])$ of order dividing 2 that sends an m-tuple in (72) to $\mathbf{e}_3(\phi[r]) \in H^3(K)$.

Let $r \in \bar{R}''_1 + \bar{R}''_2$, where $\bar{R}''_1 = \langle \bar{e}_i \in R' \rangle$ and $\bar{R}''_2 = \langle \bar{e}_j + \bar{e}_k \in R' \mid \bar{e}_j, \bar{e}_k \notin R', n_j \equiv n_k \equiv 1 \mod 2 \rangle$. Then, by (65) both invariants $\mathbf{e}_3(\phi[\bar{e}_i])$ and $\mathbf{e}_3(\phi[\bar{e}_j + \bar{e}_k])$ vanish for any $\bar{e}_i \in \bar{R}''_1$ and any $\bar{e}_j + \bar{e}_k \in \bar{R}''_2$, thus $\mathbf{e}_3(\phi[r])$ vanishes.

As before, by Theorem 5.6 it is enough to show that the invariant $\mathbf{e}_3(\phi[r])$ is nontrivial for any $r \in R' \backslash (\bar{R}''_1 + \bar{R}''_2)$. Let $G'_{\mathrm{red}} = (\mathbf{\Omega}_6)^m / \boldsymbol{\mu}$. Then, the same arguments as in the proof of Proposition 6.3 work if we replace [14, Lemma 4.3], the exceptional isomorphism $A_1 = B_1$, the standard embedding $\mathbf{\Gamma}_3 \to \mathbf{\Gamma}_{2n_i+1}$, and Lemma 6.4 in the proof of Proposition 6.3 by [14, Lemma 4.2], the exceptional isomorphism $A_3 = D_3$, the standard embedding $\mathbf{\Omega}_6 \to \mathbf{\Omega}_{2n_i}$, and Lemma 6.14, respectively. □

We shall present the following analogue of Lemmas 6.4 and 6.9.

Lemma 6.14. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ defined over an algebraically closed field F, where $m \geq 1$, $n_i \geq 3$, $\boldsymbol{\mu}$ is a central subgroup. Set $G_{\mathrm{red}} = (\prod_{i=1}^m \mathbf{\Omega}_{2n_i})/\boldsymbol{\mu}$. Then, every normalized invariant in $\mathrm{Inv}^3(G_{\mathrm{red}})$ is ramified if either $n_i \geq 4$ for some $i \in I_1 \cup I_2$ or $n_j n_k \not\equiv 1 \mod 2$ for some j and k such that $\bar{e}_j + \bar{e}_k \in R'$.*

Proof. Let α be a normalized invariant in $\operatorname{Inv}^3(G_{\mathrm{red}})$ be written as in (75) for some subsets $I_1' \subseteq I_1$, $I_2' \subseteq I_2$ and $R'' \subseteq R'$.

First, assume that there exists $j \in I_1'$. Let $Q = (x, y)$ be a division quaternion algebra over a field extension K/F and let $\psi_j = \langle\langle x, y, z\rangle\rangle \perp h$ be a quadratic form over $E := K((z))$, where h denotes a hyperbolic form. Choose a G_{red}-torsor $\eta = \big((A_1, \sigma_1, f_1), \dots, (A_m, \sigma_m, f_m)\big)$ such that

$$(A_j, \sigma_j, f_j) = (M_{2n_j}(E), \sigma_{\psi_j}) \text{ and } (A_i, \sigma_i, f_i) = (M_{2n_i}(E), t)$$

for all $1 \leq i \neq j \leq m$, where σ_{ψ_j} denotes the adjoint involution on $M_{2n_j}(E)$ with respect to ψ_j and t denotes the transpose involution on $M_{2n_i}(E)$. Then, we have

$$\sum_{i\in I_1'} \mathbf{e}_{3,i}(\eta) = (x, y, z),\ \sum_{i\in I_2'} \Delta_i'(\eta) = \sum_{r\in R''} \mathbf{e}_3(\phi[r])(\eta) = 0.$$

Therefore, we have $\partial_z(\alpha(\eta)) = (x, y) \neq 0$. Hence, the invariant α ramifies.

We assume that $I_1' = \emptyset$ and $I_2' \neq \emptyset$, i.e., $\alpha(\eta) = \sum_{i\in I_2'} \Delta_i' + \sum_{r\in R''} \mathbf{e}_3(\phi[r])$. Let $j \in I_2'$ and let $\eta = \big((A_1, \sigma_1, f_1), \dots, (A_m, \sigma_m, f_m)\big)$ be a G_{red}-torsor over L as in the proof of Proposition 6.13. Then, over $L((y_j))$ we have

$$\partial_{y_j}\big(\alpha(\eta)\big) = \partial_{y_j}\big(\Delta_j'(\eta)\big) = \partial_{y_j}\big((x_{j,1}, x_{j,2}, y_j)\big) = (x_{j,1}, x_{j,2}) \neq 0,$$

thus the invariant α ramifies.

Now we may assume that $n_i \not\equiv 0 \mod 4$ and $R_{1,i}'$, $R_{1,i} \neq Z_i$ for all $1 \leq i \leq m$, thus

$$\alpha(\eta) = \mathbf{e}_3(\phi[r_2]) + \mathbf{e}_3(\phi[r_3])$$

for some nonzero $r_2 \in R_2$ and $r_3 \in R_3$, where R_2 denotes the subspace of $\bar{R}$ generated by $\bar{e}_i + \bar{e}_j$ for all $1 \leq i \neq j \leq m$, R_3 denotes a complementary subspace of R_2 in $\bar{R}$, and η is a G_{red}-torsor. For simplicity, we write $e(i_1, \dots, i_k)$ for $\bar{e}_{i_1} + \cdots + \bar{e}_{i_k}$. Choose bases $B_2 = \{e(i_p, i_{p,q})\}$ of R_2 with $n_{i_{p,q}} \geq n_{i_p}$ and B_3 of a complementary subspace of R_2 as in Lemma 6.4 so that the invariant α is written as in (51).

To show that the invariant $\alpha(\eta)$ ramifies, we now proceed as in the proof of Lemma 6.9, with the following simple modifications. Let (Q, γ), (Q_1, γ_1), (Q_2, γ_2) be the quaternions with canonical involutions as in the proof of Lemma 6.9 and let σ be an orthogonal involution on Q given by the composition of γ and the inner automorphism induced by one of the generators of pure quaternions in Q. Then, the same proof as in Lemma 6.9 still works if we choose $\eta = \big((A_i, \sigma_i, f_i)\big)$ satisfying (66), (67), (68) for Case 1 and (69), (70), and (71) for Case 2, after replacing the involutions γ_1', γ, and σ_ω in those equations by γ_1, σ, and t, respectively. □

Finally, we prove the second main result on the group of unramified degree 3 invariants for type D.

Theorem 6.15. *Let $G = (\prod_{i=1}^m \mathbf{Spin}_{2n_i})/\boldsymbol{\mu}$ defined over an algebraically closed field F, $m \geq 1$, $n_i \geq 3$, where $\boldsymbol{\mu}$ is a central subgroup. Then, every unramified degree 3 invariant of G is trivial, i.e., $\operatorname{Inv}^3_{\mathrm{nr}}(G) = 0$.*

Proof. Let $G_{\mathrm{red}} = (\prod_{i=1}^{m} \mathbf{\Omega}_{2n_i})/\boldsymbol{\mu}$, $G'_{\mathrm{red}} = (\mathbf{\Omega}_6)^m/\boldsymbol{\mu}$, and $G' = (\mathbf{SL}_4)^m/\boldsymbol{\mu}$. Then, by the same argument as in the proof of Theorem 6.5 together with Proposition 6.13 and Lemma 6.14 we may assume that the bottom map in (57) is an isomorphism. By [14, Lemma 4.2], we have $\mathrm{Inv}^3_{\mathrm{nr}}(G'_{\mathrm{red}}) = 0$. Hence, every invariant of G_{red} is ramified. $\square$

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