A GENERALIZATION OF THE AZ IDENTITY

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Abstract. The identity discovered in [1] can be viewed as a sharpening of the LYM inequality ([3], [4], [5]). It was extended in [2] so that it covers also Bollobas' inequality [6]. Here we present a further generalization and demonstrate that it shares with its predecessors the usefullness for uniqueness proofs in extremal set theory.

1. Introduction

A few years ago Ahlswede and Zhang [1] found the following identity.

Theorem AZ_1 **.** *For every family* $\mathcal{A} \subset 2^\Omega$ *of non–empty subsets of* $\Omega = \{1, 2, \ldots, n\}$

$$\sum_{X \subset \Omega} \frac{W_\mathcal{A}(X)}{|X| \binom{n}{|X|}} = 1 \ , \quad \textit{where} \ \ W_\mathcal{A}(X) = \left| \bigcap_{X \supset A \in \mathcal{A}} A \right| \ .$$

We associate with every $\mathcal{E} \subset 2^\Omega$ the upset $\mathcal{U}(\mathcal{E}) = \{U \subset \Omega : U \supset E$ for some $E \in \mathcal{E}\}$ and the downset $\mathcal{D}(\mathcal{E}) = \{D \subset \Omega : D \subset E$ for some $E \in \mathcal{E}\}$.

When $\mathcal{A}$ is an antichain in the poset $(2^\Omega, \supset)$, then the identity becomes

$$\sum_{X \in \mathcal{A}} \frac{1}{\binom{n}{|X|}} + \sum_{X \in \mathcal{U}(\mathcal{A}) \smallsetminus \mathcal{A}} \frac{W_\mathcal{A}(X)}{|X| \binom{n}{|X|}} = 1 \ . \tag{1}$$

The LYM inequality is obtained by omission of the second summand, which by definition of $W_\mathcal{A}$ can also be written in the form $\sum_{X \notin \mathcal{D}(\mathcal{A})} \frac{W_\mathcal{A}(X)}{|X| \binom{n}{|X|}}$. We call this the deficiency of the inequality.

More generally, in [2] the Bollobas inequality was lifted to an identity.

Theorem AZ_2 **.** *For two families* $\mathcal{A} = \{A_1, \ldots, A_N\}$ *and* $\mathcal{B} = \{B_1, \ldots, B_N\}$ *of subsets of* Ω *with the properties*

(a) $A_i \subset B_i$ *for* $i = 1, 2, \ldots, N$
(b) $A_i \not\subset B_j$ *for* $i \neq j$

$$\sum_{i=1}^{N} \frac{1}{\binom{n-|B_i \smallsetminus A_i|}{|A_i|}} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}} = 1 \ . \tag{2}$$

In [1] it was explained that Theorem AZ_1 gives immediately, what LYM does not, namely the uniqueness part in Sperner's Theorem. In [2] the uniqueness of an optimal configuration of unrelated chains of subsets due to Griggs, Stahl and Trotter [7] was proved with the help of Theorem AZ_2 .

Recently, Körner and Simonyi [10] observed the LYM–type inequality:

For $\mathcal{A} = \{A_1, \dots, A_N\}, \mathcal{B} = \{B_1, \dots, B_N\} \subset 2^\Omega$ with

$$A_i \cap B_i = \varnothing, A_i \not\subset A_j \cup B_j, B_i \not\subset A_j \cup B_j \text{ for } i \neq j$$

$$\sum_{i=1}^{N} \binom{n-|A_i|}{|B_i|}^{-1} + \binom{n-|B_i|}{|A_i|}^{-1} - \binom{n}{|A_i|+|B_i|}^{-1} \leq 1 \tag{3}$$

and they asked (Problem 2) "Is this inequality ever tight?".

This rather modest question was a challenging test of the power of the identities in [1], [2] or, more precisely, of the procedure to produce new identities described in [1].

The outcome is an Ahlswede–Zhang type identity (Theorem 1) which goes considerably beyong Theorem AZ_2 . From a special case of this identity we derive <u>a full characterization</u> of the cases with equality (Theorem 2) even for a generalized version of (3). In other words we characterize the cases with deficiency zero.

2. The identity

Theorem 1. *Suppose that for a family $\mathcal{B} = \{B_1, \dots, B_N\}$ of subsets of Ω and a family $\mathcal{A}^* = \{\mathcal{A}_1, \dots, \mathcal{A}_N\}$ of subsets of 2^Ω , where $\mathcal{A}_i = \{A_i^t : t \in T_i\}$ for a finite index set T_i , we have the properties*

(a) $A_i^t \subset B_i$ *for* $t \in T_i$ *and* $i = 1, 2, \dots, N$

(b) $A_i^t \not\subset B_j$ *for* $t \in T_i$ *and* $i \neq j$.

Then with $\mathcal{A} = \bigcup_{i=1}^N \mathcal{A}_i$

$$\sum_{i=1}^{N}\sum_{k=1}^{|T_i|}(-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n-|B_i - \bigcup_{t\in S} A_i^t|}{|\bigcup_{t \in S} A_i^t|}^{-1} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}} = 1 \ . \tag{4}$$

The specialisation $|T_i| = 1$ for $i = 1, \dots, N$ gives Theorem AZ_2 . The proof goes again by counting chains. A key tool in [2] was