THE MAXIMAL LENGTH OF CLOUD–ANTICHAINS

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1. Introduction

The notion of an antichain in a partially ordered set was generalized [2] and [3] to the seemingly natural notion of a "cloud–antichain" $\{\mathcal{A}_i\}_{i=1}^N$. Whereas in antichains *elements* of a partially ordered set are compared in cloud–antichains *sets of elements* take their role. Elements in different sets $\mathcal{A}_i$, called clouds, are required to be incomparable. Formally, for every two clouds $\mathcal{A}_i$ and $\mathcal{A}_j$ we have

$$A_i \not> A_j \text{ for } \textit{all } A_i \in \mathcal{A}_i \text{ and } \textit{all } A_j \in \mathcal{A}_j. \tag{1.1}$$

In [3] further notions of cloud–antichains were introduced. Whereas the logical structure of the formula (1.1) suggests to speak of an antichain of *type* $\underline{(\forall, \forall)}$, the new notions in [3] are of the types $(\forall, \exists)$, $(\exists, \forall)$, and $(\exists, \exists)$.

In the sequal we consider always the partially ordered set $\mathcal{P} = 2^{\Omega_n}$, the power set of $\Omega_n = \{1, 2, \dots, n\}$, with set theoretic containment as order relation. $\{\mathcal{A}_i\}_{i=1}^N$ is always a family of subsets of $\mathcal{P}$. It is said to be of type $(\exists, \forall)$, if for all $i \neq j$

$$\text{there } \textit{exists} \text{ an } A_i \in \mathcal{A}_i \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \text{ for } \textit{all } A_j \in \mathcal{A}_j \ , \tag{1.2}$$

it is of type $(\forall, \exists)$, if for all $i \neq j$

$$\text{for } \textit{all } A_i \in \mathcal{A}_i \text{ there } \textit{exists} \text{ an } A_j \in \mathcal{A}_j \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \tag{1.3}$$

and it is of type $(\exists, \exists)$, if for all $i \neq j$ there *exists* an $A_i \in \mathcal{A}_i$

$$\text{and there } \textit{exists} \text{ an } A_j \in \mathcal{A}_j \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \ . \tag{1.4}$$

The maximal cardinalities N of such systems as functions of n are denoted by $N_n(\exists, \forall)$, $N_n(\forall, \exists)$, and $N_n(\exists, \exists)$, resp.

Obviously, an analogously defined quantity $N_n(\forall, \forall)$ equals $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, because in an optimal configuration $|\mathcal{A}_i| = 1$ and Sperner's classical Theorem ([1]) applies. We also study systems with *disjoint* clouds. The maximal cardinalities are then denoted by $M_n(\exists, \forall)$, $M_n(\forall, \exists)$, and $M_n(\exists, \exists)$, resp.

We call two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ asymptotically equivalent and write $f(n) \sim g(n)$, if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$