Primer on CPP Geometry

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Our purpose is to describe some basic ideas and constructions in C^1, piecewise–projective (CPP for short) geometry on the circle. This geometry seems to provide a good context in which to discuss (see definition 1.1), the Higman–Thompson group G (see [Br], [GhS]) and certain of its subgroups (see [G2]).

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0.1 Introduction

We require some notation. Consider the action of $\mathrm{PSL}_2\ \mathbb{R}$ on $\mathbb{H}$, the hyperbolic disk, by linear fractional transformations. The action extends to a transitive action on $S^1 = \mathbb{R} \cup \{\infty\}$. Let, for $x \in S^1$, A_x (respectively P_x) denote the subgroup fixing x (resp. fixing x and with derivative 1 at x). The elements of P_x are called parabolic, and

$$\begin{aligned} A_\infty &= \left\{ \begin{pmatrix} r & \frac{b}{r} \\ 0 & \frac{1}{r} \end{pmatrix} : r > 0, b \in \mathbb{R} \right\} = \{x \longmapsto r^2 x + b\} \\ P_\infty &= \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \{x \longmapsto x + t\} \simeq \mathbb{R}. \end{aligned}$$

The circle is given the usual counterclockwise orientation.

Definition 0.1.1 *A homeomorphism $g : V \longrightarrow \mho$ between open subsets of S^1 is C^1, piecewise–projective (we write $g \in CPP$) if g has a continuous derivative, and if there is a discrete subset S of V such that for every connected component of $V - S$, $g|_C \equiv g_c$, where $g_c \in \mathrm{PSL}_2\ \mathbb{R}$. Denote by $CPP(S^1)$ the group of CPP homeomorphismus of S^1.*

If $g \in CPP$, $g : V \longrightarrow \mho$ and $x \in V$, then g^l_x, g^r_x, the <u>left</u> and <u>right germs of g at x</u>, are the elements of $\mathrm{PSL}_2\ \mathbb{R}$ such that $g|_{(x-\varepsilon,x)} \equiv g^l_x$, $g|_{(x,x+\varepsilon)} \equiv g^r_x$, for some small ε. Define the <u>change</u> at x by $D_x g \equiv (g^r_x)^{-1} g^l_x$. Since g is C^1, $D_x g \in P_x$ for all x. The set $bk(g)$ of <u>breakpoints</u> of g is the set of x such that $D_x g \neq id$. Note that $bk(g)$ is a finite set if $g \in CPP(S^1)$.

Proposition 0.1.1.1 *Let $bk(g) = \{x_1, \ldots, x_n\}$ in counterclockwise order. Then $D_{x_n} g \circ \ldots \circ D_{x_1} g = id$.*

Proof: For all i, $1 \leq i < n$, we have $g^r_{x_i} = g^l_{x_{i+1}}$, and $g^r_{x_n} = g^l_{x_1}$.

The result follows.