Piecewise projective and piecewise Steinberg homomorphisms, and a formula in H_1

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1 Introduction

Let $\mathrm{CPP}_C(S^1)$ denote the group of C^1, piecewise $\mathrm{SL}_2\mathbb{R}$ homomorphisms of $S^1 = \mathbb{R}^2 - (0,0)/\mathbb{R}^+$ which are the identity near the ray through $(0,1)$ (see 2.1). A remarkable feature of this group ([G2]) is that its abelianization is isomorphic to $H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R});\mathbb{Z})$, where $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal cover of the Lie group $\mathrm{SL}(2,\mathbb{R})$, considered as a discrete group. Our purpose in this report is to give an explicit formula: given $g \in \mathrm{CPP}_C(S^1)$ we describe an explicit $w(g) \in H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R});\mathbb{Z})$.

The isomorphism $H_1(\mathrm{CPP}_C(S^1)) \to H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R}))$ of [G2] has the following geometric description. Let $g \in \mathrm{CPP}_C(S^1)$, and consider the mapping torus of g, with its codimension–one foliation whose leaves contain the orbits of g. Since g is supported away from $(0,1)$, we can cut at $(0,1)$ to obtain a mapping sphere S^2_g (see [Ma]) with a codimension–one foliation whose leaves are cicles near the poles. This transversely CPP singular foliation determines a map from S^2_g to BCPP, the Haefliger classifying space ([H]) for such structures, and thus an element $[S^2_g] \in H_2(\mathrm{BCPP})$. Let $|\mathrm{SL}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}|$ denote the flat S^1 bundle over a $K(\mathrm{SL}(2)\mathbb{R}),\nVdash)$; this space is a $K(\widetilde{\mathrm{SL}}(2,\mathbb{R}),\nVdash)$ and possesses a natural map $\varphi : |\mathrm{SL}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}| \to \mathrm{BCPP}$ which induces an isomorphism in homology ([G2]) and so $[S^2_g] = \varphi_*(\chi_g)$, $\chi_g \in H_2|\mathrm{SL}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}| = \mathbb{H}_{\nvDash}(\widetilde{\mathrm{SL}}(\nvDash,\mathbb{R});\mathbb{Z})$. How can we describe χ_g derectly, in terms of g?

Recall ([A],[Mi]) that $H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R})) \simeq \mathrm{div}(\mathbb{K}_{\nvDash}\mathbb{R})$, the divisible part of $K_2\mathbb{R}$. This group is isomorphic to $\mathbb{R}^+ \wedge \mathbb{R}^+/\diagdown\wedge\nVdash-\diagdown$, the exterior product of $\mathbb{R}^+$ with itself over $\mathbb{Z}$, modulo the subgroup generated by $r \wedge 1-r$, $0<r<1$. Given $g \in \mathrm{CPP}_C(S^1)$, we define an element $w(g) \in K_2\mathbb{R}$ as follows: Let $\chi_1, \ldots, \chi_n \in S^1$, listed in counterclockwise order, be the points where the germs $g^l_{\chi_i}, g^r_{\chi_i} \in \mathrm{SL}(2,\mathbb{R})$ (see 2.1) of g just clockwise and counterclockwise of χ_i differ, and set $D_{\chi_i}g = (g^r_{\chi_i})^{-1}g^l_{\chi_i}$. Clearly, we have

$$\mathrm{id} = D_{\chi_n}g \ldots D_{\chi_1}g \tag{1}$$

in $\mathrm{SL}(2,\mathbb{R})$. Since g is C^1, each $D_{\chi_i}g$ is parabolic, and may be expressed as

$$D_{\chi_i}g = \begin{pmatrix} 1 & 0 \\ -a_i & 1 \end{pmatrix} \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_i & 1 \end{pmatrix} \qquad a_i, b_i \in \mathbb{R}$$

(or simply as $D_{\chi_i}g = \begin{pmatrix} 1 & 0 \\ -c_i & 1 \end{pmatrix}$ if $\chi_i = (0,-1)$). Recall ([Mi],[A]) the generators of the Steinberg group $\chi_{12}(\lambda), \chi_{21}(\mu)$, $\lambda, \mu \in \mathbb{R}$ which map to $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \in$ $\mathrm{SL}(2,\mathbb{R})$, and set

$$y_i = \begin{cases} x_{21}(-a_i)x_{12}(b_i)x_{21}(a_i) & chi_i \neq (0,-1) \\ x_{21}(c_i) & chi_i = (0,-1) \end{cases}$$

By equation (1,1), $y_n \dots y_1$ lies in the kernel of the projection $\mathrm{St}(2,\mathbb{R}) \to \mathrm{SL}(\nvDash,\mathbb{R})$, and so $w(g) \equiv y_n \dots y_1 \in K_2\mathbb{R}$.

Theorem 1.1 *$w(g) \in divK_2\mathbb{R}$, and $\varphi(\precapprox(\eth)) = [\mathbb{S}^{\nvDash}_{\eth}]$.*

In section 2, the definition of a group of "piecewise Steinberg" homeomorphisms of S^1 implies that w is a homomorphism. In section 3, theorem 1.1 is proved by calculating $w(g)$ and $[S^2_g]$ for a family of g which generates $\mathrm{CPP}_C(S^1)$. A different approach in section 4 shows directly that w is an isomorphism $H_1\mathrm{CPP}_C(S^1) \to$div $K_2\mathbb{R}$. Sundry remarks conclude the paper.

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2 Piecewise projective and piecewise Steinberg groups

We regard $\mathrm{SL}(2,\mathbb{R})$ as acting on the circle of rays $S^1 = \mathbb{R}^{\nvDash} - (\nvdash,\nvdash)/\mathbb{R}^+$; we will abuse notation and denote points in S^1 by ordered pairs, so that $(1,0) = (2,0)$, for example.

Definition 2.1 *Let $CPP_C(S^1)$ denote the group of C^1 homeomorphisms of S^1, supported away from $(0,1)$, such that there is a finite set $S \subset S^1$ such that for every connected component C of S^1–S, $g|_C \equiv g_C \in SL(2,\mathbb{R})$. If $s \in S^1$, $g \in CPP_C(S^1)$ let $g^l_s, g^r_s \in SL(2,\mathbb{R})$ be the restrictions of g to small neighborehoods clockwise and counterclockwise of s, and set $D_sg = (g^r_s)^{-1}g^l_s$.*