The Pfaff–Saalschütz identity in hyperbolic geometry

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1 Introduction

Let $P_n(\mathbb{C})$ denote the complex vector space of all polynomials in z and $\bar{z}$ whose degree at z and at $\bar{z}$ is bounded by n. Let $P_n(\mathbb{H}^3)$ be the complex vector space of all polynomials in $z, \bar{z}$ and t whose total degree is less or equal to n. Both spaces are natural $PSL_2(\mathbb{C})$–modules: $P_n(\mathbb{C})$ gets its $PSL_2(\mathbb{C})$ structure from the canonical action of $SL_2(\mathbb{C})$ on $\mathbb{C}^2$, $P_n(\mathbb{H}^3)$ is acted on by $PSL_2(\mathbb{C})$ as a submodule of the ring of affine Klein coordinates on three dimensional hyperbolic space, see line (2) below.

$P_n(\mathbb{C})$ is irreducible. $P_2(\mathbb{H}^3)$ contains a non zero element which is invariant under $PSL_2(\mathbb{C})$, namely $r^2 := t - z\bar{z}$. This statement is commonly known as one of the exceptional isomorphisms between Lie groups. It identifies $PSL_2(\mathbb{C})$ and $SO(1,3)$.

There is a natural homomorphism

$$h_n : P_n(\mathbb{H}^3) \rightarrow P_n(\mathbb{C}).$$

It is defined by

$$h_n(t) = z\bar{z}$$

and it respects the action of $PSL_2(\mathbb{C})$ on both modules. It geometrically means restricting functions defined on a three dimensional ball to its boundary which is a two dimensional sphere. Here the interior of the ball plays the role of real Lobačevskij (or, equivalently, hyperbolic) space in dimension 3, and its boundary is identified with the Riemann sphere.

We, by explicitly exhibiting a homomorphism η_n inverse to h_n:

$$\begin{aligned} \eta_n : P_n(\mathbb{C}) &\rightarrow P_n(\mathbb{H}^3) \\ h_n \circ \eta_n &= idP_n(\mathbb{C}) \end{aligned}$$

show that h_n is surjective. η_n, of course, will be $PSL_2(\mathbb{C})$–equivariant, again. It will be uniquely determined by these properties, so $P_n(\mathbb{H}^3)$ contains $P_n(\mathbb{C})$ as an irreducible

$PSL_2(\mathbb{C})$–module exactly once. Surprisingly, the equivariance of η_n is equivalent to the validity of the so called Pfaff–Saalschütz identity. The kernel of h_n can easily be identified with $P_{n-2}(\mathbb{H}^3)$, so we get a decomposition

$$P_n(\mathbb{H}^3) = \bigoplus_{2j \leq n} P_{n-2j}(\mathbb{C})$$

into irreducible submodules.

We finally observe that η_n relates the Laplacian Δ_E on $\mathbb{C}$ as a Euclidean plane to the Laplacian Δ_L on three dimensional Lobačevskij space $\mathbb{H}^3$.

It seems to be worthwile studying algebraic geometric objects like the set of zeros of some $\varphi \in P_n(\mathbb{H}^3)$. If, in particular, φ has the form $\eta_n(t)$ for some $f \in P_n(\mathbb{C})$ then its set of zeros in $\mathbb{H}^3$ will be a geometric object which perpendicularly meets the boundary, its "trace" there being the set of zeros of $f(z, \bar{z})$ on the Riemann sphere. This observation comprises and generalizes well known facts from (linear) elementary hyperbolic geometry in dimension three, as, e.g., compiled in [2]. An example will be discussed.

2 Prerequisits

We collect a few facts from elementary geometry. Let $\mathcal{H}_2$ be the collection of all hermitean forms on $\mathbb{C}^2$. It is a fourdimensional real vector space. Taking Gram matrices

$$Gr(\beta) = \begin{pmatrix} \beta(e_1, e_1) & \beta(e_1, e_2) \\ \beta(e_2, e_1) & \beta(e_2, e_2) \end{pmatrix}$$

for $\beta \in \mathcal{H}_2$, with respect to some fixed basis (e_1, e_2) of $\mathbb{C}^2$, identifies $\mathcal{H}_2$ and

$$\left\{ \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} | \xi, \eta \in \mathbb{R}, \zeta \in \mathbb{C} \right\}.$$

$\mathcal{H}_2$ carries a geometry $\langle \quad , \quad \rangle$:

$$\langle \beta, \beta \rangle = det Gr(\beta)$$

is a quadratic form on $\mathcal{H}_2$ with signature (1,3).

The group $PSL_2(\mathbb{C})$ acts, in a canonical way, on $\mathcal{H}_2$, which is expressed explicitly by

$$\begin{aligned} \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} & \rightarrow S \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} {}^t\bar{S} \\ & = \begin{pmatrix} a\bar{a}\xi + a\bar{b}\zeta + \bar{a}b\bar{\zeta} + b\bar{b}\eta & a\bar{c}\xi + a\bar{d}\zeta + b\bar{c}\bar{\zeta} + b\bar{d}\eta \\ \bar{a}c\xi + \bar{b}c\zeta + \bar{a}d\bar{\zeta} + \bar{b}d\eta & c\bar{c}\xi + c\bar{d}\zeta + \bar{c}d\bar{\zeta} + d\bar{d}\eta \end{pmatrix} \end{aligned} \tag{1}$$