HIGHER LEVEL EXTREMAL PROBLEMS

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1. Introduction

There seems to be an ever increasing number of sometimes fascinating combinatorial extremal problems. In particular Computer Science is a rich source of such problems. Our observation is that several of the known basic problems and many new ones can be formulated as parts of a transparent scheme of problems. It gives a certain orientation in a substantial part of known combinatorial results and at the same time it gives directions for new investigations (see section 9). Our present understanding of the subject grew with our recent work, especially, the papers [17], [9], [18], [20]. We focus here on the most important poset in combinatorics, the power set $\mathcal{P}(\Omega)$ of a finite set $\Omega = \{1, 2, \dots, n\}$, and we study problems involving the following binary relations between its elements:

$$\begin{aligned} &A \supset\subset B \quad (\text{comparable, that is, } A \subset B \text{ or } A \supset B), \\ &A \supset|\subset B \quad (\text{incomparable}), \quad A \cap B = \varnothing \quad (\text{disjoint}), \\ &A \cap B \neq \varnothing \quad (\text{intersecting}). \end{aligned} \tag{1.1}$$

The following problem, due to and solved by Sperner [12], has often been cited as a prototype of an extremal problem.

$\mathcal{A} \subset \mathcal{P}(\Omega)$ is an antichain, if its members are incomparable. What is the maximal cardinality of antichains? The problem is based on the relation "incomparable".

If this relation is replaced by "intersecting" and we also impose the constraint $\mathcal{A} \subset \mathcal{P}_k(\Omega) \triangleq \binom{\Omega}{k}$, then we arrive at the Erdös/Ko/Rado problem [13]. Analogous problems for the relations "comparable" or "disjoint" turn out to be trivial.

Furthermore, in the sequel we speak of the *unresricted* case, if $\mathcal{P}(\Omega)$ is the ground set, and of the *restricted* case, if $\mathcal{P}_k(\Omega)$ is the ground set.

A multitude of problems arises, if we go one step higher, that is, from sets to families of sets etc.

To fix ideas let us recall the notion of a cloud–antichain from [9].

$\{\mathcal{A}_i : 1 \leq i \leq N\}$ with disjoint non–empty $\mathcal{A}_i \subset \mathcal{P}(\Omega)$ is a cloud–antichain (CAC), if for all $i \neq j$ and all $A_i \in \mathcal{A}_i$, $A_j \in \mathcal{A}_j$

$$A_i \supset|\subset A_j. \tag{1.2}$$

Here not only the length N but also $|\mathcal{A}_i|$ $(1 \leq i \leq N)$ are parameters of interest. The case $\max\limits_i |\mathcal{A}_i| = 1$ is Sperner's case.

Notice that for every (i,j) we require (1.2) for *every* $A_i \in \mathcal{A}_i$ and *every* $A_j \in \mathcal{A}_j$. We therefore refer to this problem as being of *type* $(\forall, \forall)$.

A weaker requirement is that there *exists* an $A_i \in \mathcal{A}_i$ such that for *all* $A_j \in \mathcal{A}_j$ (1.2) holds. It is called of type $(\exists, \forall)$. An even weaker condition is that for *all* $A_i \in \mathcal{A}_i$ there *exists* an $A_j \in \mathcal{A}_j$ with the property (1.2). We speak of type $(\forall, \exists)$. Finally, the weakest condition is the type $(\exists, \exists)$, where again for every (i,j) there *exists* $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ with (1.2).