The Classification of Finitely Spreading Graphs

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Thomassen recently introduced the concept of a finitely spreading graph: an infinite graph whose edges can be oriented, each in one, both, or neither direction, so that every vertex has finite out-degree and every ray has a forward oriented tail. He conjectured that a graph is finitely spreading if and only if it is bounded—equivalently, if it contains none of three specified infinitely spreading graphs. We prove Thomassen's conjecture in amended form, adding a fourth minimal obstruction to the three conjectured ones.

Introduction

There are two basic structural phenomena that make an infinite graph 'truly infinite': vertices of infinite degree, and infinite paths. These features are in a sense dual: one or the other occurs in every infinite connected graph. If only one occurs, i.e. if the graph is rayless or locally finite, its structure tends to be much simpler and easier to describe. This makes it desirable to identify, if possible, in an arbitrary infinite graph a (say) locally finite substructure that captures the essence of certain aspects of the structure of the whole graph. An important such aspect for an infinite graph, as suggested above, lies in its rays—whose 'essence' is their *eventual* route, their behaviour at infinity.

The subject of this paper is a conjecture of Thomassen, which determines when the rays in an infinite graph eventually follow a certain locally finite substructure. We call a graph finitely spreading if its edges can be oriented, each in one, both, or neither direction, so that every vertex has finite out-degree and every ray has a forward oriented tail.

More precisely, we wish to assign to some of the edges xy one or both of their two possible orientations $x \to y$ and $x \leftarrow y$. If the orientation $x \to y$ has been assigned to xy, we say that xy is oriented *from x to y*, or *forward*; note that this does not preclude the possibility that xy has the orientation $x \leftarrow y$ assigned to it as well. (The expression 'forward' here is a reference not to the edge itself but to our present notation of it as xy rather than yx.) The aim, then, is twofold. First, every vertex x should have only finitely many neighbours y such that the edge xy is oriented from x to y. Second, every ray (one-way infinite path) $x_0x_1\dots$ should have a tail (subray) $x_nx_{n+1}\dots$ such that every edge $x_{n+i}x_{n+i+1}$ of this tail is oriented forward; such a tail will be called a *forward oriented tail*. If it is possible to orient the edges of a graph in this way, the graph is called *finitely spreading*, and its orientation a *finitely spreading orientation*; if not, the graph is *infinitely spreading*.

Thus, a finitely spreading orientation provides a graph with a locally finite substructure mapping out the essential directions of its rays: every ray will eventually coincide with a ray indicated by the orientation.

Before we state Thomassen's conjecture, let us see some examples. Any locally finite graph is trivially finitely spreading: just orient every edge both ways. Let us remark at this point that the provision for two-way orientations of edges is necessary if we want the locally finite graphs to be included among the finitely spreading ones (as seems clearly desirable): the infinite ladder is a simple example of a locally finite graph which has no finitely spreading orientation using only one-way orientations of edges.

The regular tree of countably infinite degree, which we denote by T_ω, is a simple example of an infinitely spreading graph. However, a graph need not be 'wide' to be infinitely spreading: a 'long and narrow' example is given by the graph B_0 shown in Fig. 1. (Formally, B_0 is obtained from a ray $x_0 x_1 \dots$ by adding, for each $n \in \mathbb{N}$, a countably infinite set of independent x_{3n+1}–x_{3n+3} paths, disjoint for different n.)

FIGURE 1. The 'bundle graph' B_0

It is clear that B_0 does not admit a finitely spreading orientation: each of the vertices x_{3n+1} would be incident with an edge on its right not oriented away from it, and one would readily find a ray through B_0 using all these edges from left to right, i.e. against their possible orientation.

The graph F_0 of Fig. 2, obtained from a ray $R = x_0 x_1 \dots$ by adding disjoint rays $Q_2, Q_4, Q_6, \dots$ with $Q_k \cap R = \{ x_k \}$, and joining x_{2n+1} to all the new vertices of Q_{2n+2} for every $n \in \mathbb{N}$, is a similar example of an infinitely spreading graph. Interestingly, this graph becomes finitely spreading (and our first non-trivial example of such a graph) if its 'fans' are flipped horizontally, i.e. if the vertices x_{2n+1} are joined to all the vertices of Q_{2n} rather than to those of Q_{2n+2} (add a ray Q_0).

We are now ready to state Thomassen's conjecture. In its original form, the conjecture says that a countable graph is finitely spreading if and only if it is bounded; a graph is called *bounded* if for every labelling of its vertices with natural numbers there exists an infinite sequence of natural numbers which eventually exceeds the labelling along any ray in the graph. (A more formal definition of boundedness can be found in [2] or in [3].) It is not difficult to prove that a finitely spreading countable graph must indeed be bounded [3]; the conjecture's claim is that bounded graphs are finitely spreading.