Domination Games on Infinite Graphs

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We consider two infinite games, played on a countable graph G given with an integer vertex labelling. One player seeks to construct a ray (a one-way infinite path) in G, so that the ray's labels dominate or elude domination by an integer sequence being constructed by another player. For each game, we give a structural characterization of the graphs on which one player or the other can win, providing explicit winning strategies. This answers a question of Deuber. [1]

1. Introduction

ray

Let G be a countable graph, and let $\ell\colon V(G) \to \omega$ be an injective labelling of its vertices with natural numbers (thus distinct vertices have distinct labels). Consider the following game for two players, Adam and Eve. The two players move alternately, ω times, Eve having the first move. When Eve moves, she plays a natural number; when Adam moves, he plays a vertex of G. In the course of one game, Eve thus creates a sequence $e\colon \omega \to \omega$, $n \mapsto e_n$, of numbers, while Adam creates a sequence $a\colon \omega \to V(G)$, $n \mapsto a_n$, of vertices. More specifically, Adam tries to choose his vertices so as to define a *ray* (a one-way infinite path) in G, i.e. so that a_{n+1} is adjacent to a_n for every $n \in \omega$ and no vertices are repeated.

dominating game

bounding game

Let us say that a sequence $(a_n)_{n\in\omega}$ of natural numbers *dominates* another such sequence $(b_n)_{n\in\omega}$ if $a_n \geqslant b_n$ for all n greater than some $n_0 \in \omega$. In the *dominating game*, Adam wins if his function a defines a ray whose sequence $(\ell(a_n))$ of labels dominates Eve's sequence (e_n). Otherwise, Eve wins. In the *bounding game*, Adam merely tries to escape domination by Eve: he wins iff he constructs a ray whose labels exceed the corresponding terms of Eve's sequence again and again, i.e. such that (e_n) does not dominate $(\ell(a_n))$. Note that if Eve has won a particular instance of the bounding game, this same game would also go to her credit if viewed as an instance of the dominating game. Similarly, if Adam can win the dominating game on a particular graph, he can trivially also win the bounding game on that graph.

T_ω

Let us look at a few examples. It is easy to see that Adam has a winning strategy for the dominating game on T_ω, the tree in which every vertex has countably infinite degree. Indeed, he can always find a new vertex a_{n+1} adjacent to his previous vertex a_n and such that $\ell(a_{n+1}) > e_{n+1}$. This, of course, also gives him a winning strategy for any graph that contains a copy of T_ω as a subgraph.

In Section 2, we shall prove that this simple example already exhausts Adam's resources for the dominating game. More precisely, we shall show that Adam has a winning strategy for the dominating game on G if and only if $T_\omega \subset G$, and that otherwise Eve has a winning strategy. The proof of this result will be fairly straightforward.

In contrast, the situation for the bounding game is rather more interesting. Again, Adam clearly has a winning strategy on T_ω. But more is true: since Adam only needs to beat Eve's sequence again and again, he will also win on any subdivision of T_ω, i.e. on any graph obtained from T_ω by replacing its edges with non-trivial paths whose interiors are pairwise disjoint. Any such graph will be called a $\mathrm{T}T_\omega$. $\mathrm{T}T_\omega$

FIGURE 1. The prototype bundle graph B

For similar reasons, Adam has a winning strategy on the graphs B and F shown in Figs. 1 and 2. Indeed, he simply constructs a ray from left to right, starting at the leftmost vertex (of B or F, respectively). Provided he never moves back towards the left on B or vertically up on F, he will again and again find himself at a vertex with an infinite choice of neighbours to the right: as all these neighbours carry different labels, he can choose one whose label exceeds the corresponding number just played by Eve. As before, Adam's winning strategies for B and F extend to subdivisions of these graphs; subdivisions of B will be called *bundle graphs*, subdivisions of F are *fan graphs*. *bundle graphs* *fan graphs*

FIGURE 2. The prototype fan graph F

In Section 3, we shall show that these three types of subgraph provide the unique discriminator between the graphs where Adam has a winning strategy and those where Eve can win. In other words, we shall prove that