ON SPARSE APPROXIMATIONS TO RANDOMIZED STRATEGIES AND CONVEX COMBINATIONS

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Abstract. A randomized strategy or a convex combination may be represented by a probability vector $p = (p_1, \ldots, p_m)$. p is called sparse if it has only few positive entries. This paper presents an Approximation Lemma and applies it to matrix games, linear programming, computer chess, and uniform sampling spaces. In all cases arbitrary probability vectors can be substituted by sparse ones (with only logarithmically many positive entries) without loosing too much performance.

Short running title: Sparse Approximations

1. Introduction

We present the following

Approximation Lemma:

Let $A = (a_{ij})$ be an $m \times n$–matrix over the real numbers with $0 \le a_{ij} \le 1$ for $1 \le i \le m$, $1 \le j \le n$. Let $p = (p_1, \dots, p_m)$ be a probability vector, i.e., $0 \le p_i$ for all i and $\sum_{i=1}^{m} p_i = 1$, and $\varepsilon > 0$ any positive constant. Then there exists another probability vector $q = (q_1, \dots, q_m)$ with at most $k = \left\lceil \frac{\log 2n}{2\varepsilon^2} \right\rceil$ many positive coordinates q_i such that

$$\left| \sum_{i=1}^{m} p_i a_{ij} - \sum_{i=1}^{m} q_i a_{ij} \right| \le \varepsilon \qquad \text{for all} \qquad j = 1, \dots, n \; .$$

More precisely, the probability vector q can be chosen such that $q_i = \frac{k_i}{k}$ with natural numbers k_i for all $i = 1, \dots, m$. q is called an *ε–approximation* to p .

Here $\log(.)$ means 'logarithmus naturalis', and $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \ge x\}$ is the function that rounds to the next larger integer.

Observe that the bound k in the lemma depends only on the number n of columns of A and on the approximation level ε , but not on the number m of rows, the entries a_{ij} of A , and the probabilities p_i .

The paper is organized as follows: In Section 2 we illustrate some applications of the Approximation Lemma to matrix games, linear programming, computer chess, and uniform sampling spaces. Section 3 contains the proof of the lemma. In Section 4 we have a look at algorithmic aspects of constructing the sparse probability distribution q . In Section 5 we show that the bound $k = \left\lceil \frac{\log 2n}{2\varepsilon^2} \right\rceil$ is asymptotically optimal up to the multiplicative constant $4 \log 2$. Section 6 contains a discussion of miscellaneous ideas. Especially, in Paragraph 6 C) we formulate a slightly more general version of the Approximation Lemma.

2. Applications

A) Matrix Games

In the theory of zero–sum matrix games the following Theorem holds (see for instance the books [Kar 59] and [Ros 81] as references).