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A COMBINATORIAL APPROACH TO COMPLEXITY THEORY VIA ORDINAL HIERARCHIES

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§ 0 Introduction

For many mathematicians the most noble activity lies in proving theorems. It must have come as a blow for them when Gödel [Gö 31] showed that there are unprovable theorems. At the beginning they still could find some consolation in hoping that such culprits might occur in Peano arithmetics through esoteric diagonalization arguments only. Nowadays there is a wealth of most natural valid theorems which can be stated in the language of finite combinatorics but are not provable within that system.

Mathematicians understand to a certain extent how to find unprovable theorems and how to prove their unprovability within a formal system. In that sense we are relying on the classical work by Gentzen [Ge 36], Kreisel [Kr 52] and Wainer [Wa 72]. Moreover we shall apply their beautiful ideas to something which seems to be well understood, viz to well quasi orderings. This is an old concept found in Gordan [Go 1885], and Kruskal [Kr 72] correctly pointed out that it was "a frequently discovered concept". That is why we are not reinventing it and are well aware that any sequence (s_i) of specialists starting with the author must contain an arbitrary long subsequence of experts knowing more than s_0, a fact, which gives a nice theme for this paper. Leeb was one of the first dealing with structural problems of $wqo's$, which are related to this paper [Le 73]. Beautiful ideas of P. Erdös are valuable for the analysis of such phenomena occuring in all well quasi orders. For related combinatorial questions we would also like to draw the reader's attention to the beautiful paper of Nešetřil and Loebl cf [LN 91].

§ 1 How to use complexity theory

We are interested in first order statements $\forall x \exists y A(x,y)$ in the language of Peano arithmetics where A is primitive recursive. Let $g(x)$ be the smallest y satisfying $A(x,y)$. We are interested in the question of whether g is defined for every x. Let us anticipate the answer, which has been known for a long time: If g grows fast enough then the statement " g is defined everywhere" is not provable within Peano arithmetics.

In order to specify growth rates in complexity theory define a hierarchy of reference functions. There are various hierarchies available and depending on the combinatorial problems and personal taste one can make a choice. Here we concentrate on the Wainer-Grzegorczyk hierarchy, cf [Gr 53] and [Wa 72].

The first few functions are defined as follows

$$\begin{aligned} f_0(n) &= n+1 \\ f_{i+1}(n) &= f_i \circ \ldots \circ f_i(n), \text{ where the iteration is } n \text{ fold, and finally} \\ f_\omega(n) &= f_n(n) \text{ is the Ackermann function defined by diagonalization.} \end{aligned}$$

The first few levels are well known: f_0 grows like the identity, f_1 linearly, f_2 exponentially, f_3 is the tower function; f_4 is sometimes called the "wow"-function [GRS 90], ...

Using the Cantor Normal form to define fundamental sequences representing ordinals there is no difficulty extending the hierarchy up to f_{ε_0}, for instance

$$\begin{aligned} f_{\omega+i+1}(n) &:= f_{\omega+i} \circ \cdots \circ f_{\omega+i}(n) \quad n\text{-times} \\ f_{\omega+\omega}(n) &:= f_{\omega+n}(n) \quad \text{diagonalization} \\ f_{\omega^\omega}(n) &:= f_{\omega^n}(n) \quad \text{diagonalization} \\ f_{\varepsilon_0}(n) &:= f_{\omega^{\cdot^{\cdot^{\omega}}}}(n) \qquad \text{diagonalization with the } \quad \omega - \text{ tower of height } n. \end{aligned}$$

For details cf [Wa 72].

One can measure complexity with respect to these reference functions by defining

$$(*)\begin{cases} \qquad g > h & \text{iff } \lim\limits_{n\to\infty} \dfrac{h(n)}{g(n)} = 0 \\ \text{and } \; g \sim f_\alpha & \text{iff } \alpha \text{ is the smallest ordinal with } f_{\alpha+1} > g. \end{cases}$$

One should be a aware that this complexity measure is fairly insensitive to small changes but, as we shall see, it will allow rather clean-cut statements on combinatorial complexity.