Normal tree orders for infinite graphs

J.-M. Brochet and R. Diestel

A well-founded tree T defined on the vertex set of a graph G is called normal if the endvertices of any edge of G are comparable in T. We study how normal trees can be used to describe the structure of infinite graphs. In particular, we extend Jung's classical existence theorem for trees of height ω to trees of arbitrary height. Applications include a structure theorem for graphs without large complete topological minors. A number of open problems are suggested.

1. Introduction: normal spanning trees

The aim of this paper is to see how a classical and powerful structural device for the study of countable graphs, the notion of a *normal spanning tree*, can be made available more generally. The existence of such spanning trees, while trivial in the finite case (where they are better known as depth-first search trees), is in general limited to countable graphs. By generalizing the graph theoretical trees involved to order theoretical trees, a concept better suited to express uncountably long 'ends', we shall be able to extend the classical existence theorems for normal trees to arbitrary cardinalities, while retaining much of their original strength.

Throughout the paper, G will denote an arbitrary connected graph. G
Consider a tree $T \subset G$, with a root r, say. If T spans G, the choice of r imposes a partial order on the vertex set $V(G)$ of G: write $x \leqslant y$ if x lies on the unique r–y path in T. T is called a *normal* spanning tree of G if the two endvertices of any edge of G are comparable in this order.

Normal spanning trees have proved to be a most powerful tool for the study of countable graphs; see [8]–[10] for recent examples. Naturally, it is difficult to pin this strength down to a few particular properties. All the same, there are some basic facts on which most specific applications rely.

The most fundamental of these is the fact that any two vertices are separated in G by the down-closure of their infimum in the normal spanning tree. More precisely, if T is a normal spanning tree of G, and if $\lceil x \rceil$ and $\lceil y \rceil$ denote the paths in T from the root to x and to y, respectively, then

(1.1) $\lceil x \rceil \cap \lceil y \rceil$ *separates x from y in G.*

(The interesting—and non-trivial—case of this is when x and y are incomparable in T, i.e. when neither lies on the T-path from the root to the other.)

One of the most typical 'infinite' consequences of (1.1) is the fact that for every ray (1-way infinite path) $R \subset G$ there exists a ray $B \subset T$ such that $R \cap B$ is infinite. This is usually expressed as follows:

(1.2) *Every ray of G follows a branch of T.*

Together with (1.1), (1.2) implies that normal spanning trees are topologically end-faithful; see [7] for details.

If (1.1) is combined with the connecting properties of T, we obtain another useful and intuitive property: the removal of down-closures in T leaves precisely the same components in G as in T.

(1.3) *For every vertex $x \in G$, the vertex sets of the components of $G \setminus \lceil x \rceil$ coincide with those of the components of $T \setminus \lceil x \rceil$.*

In the next sections, when we come to generalize normal spanning trees, we shall seek to preserve the above three properties (or suitable analogues) whenever possible.

It is clear that not every connected graph can have a normal spanning tree: an uncountable complete graph, for example, has none. Jung [13] has characterized the graphs that admit a normal spanning tree, as follows. Call a set $U \subset V(G)$ *dispersed* if any ray can be separated from it by a finite set of vertices.

Theorem 1.4. (Jung)
G has a normal spanning tree if and only if $V(G)$ is the union of countably many dispersed sets.

Corollary 1.5. *Every countable connected graph has a normal spanning tree.*

Corollary 1.6. *If G has a normal spanning tree and H is a connected subgraph of G, then H has a normal spanning tree.*

Despite obvious strengths such as the above corollaries, Jung's characterization does not always allow one in practice to decide whether or not some particular graph has a normal spanning tree. For example, there is a result of Halin [11] that every connected graph not containing a subdivided infinite complete graph has a normal spanning tree; the proof of this theorem, based on simplicial decompositions of graphs, is fairly involved and uses only the two corollaries above, not Theorem 1.4 itself.