ON EXTREMAL SETS WITHOUT COPRIMES

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1. Definitions, formulation of problems and conjectures

We use the following notations:

$\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{P} = \{p_1, p_2, \dots\} = \{2, 3, 5, \dots\}$ denotes the set of all primes. We set

$$Q_k = \prod_{i=1}^{k} p_i. \tag{1.1}$$

For two numbers $u, v \in \mathbb{N}$ we write $(u, v) = 1$, if u and v are coprimes.

We are particularly interested in the sets

$$\mathbb{N}_s = \{u \in \mathbb{N} : (u, Q_{s-1}) = 1\} \tag{1.2}$$

and

$$\mathbb{N}_s(n) = \mathbb{N}_s \cap [1, n], \tag{1.3}$$

where for $i \le j$ $[i, j]$ equals $\{i, i+1, \dots, j\}$.

Erdös introduced in [1] (and also in [2], [4], [5]) $f(n, k, s)$ as the largest integer r for which an

$$A_n \subset \mathbb{N}_s(n), \ |A_n| = r \tag{1.4}$$

exists with no $k+1$ numbers of A_n being coprimes.

Certainly the set

$$\mathbb{E}(n, k, s) = \{u \in \mathbb{N}_s(n) : u = p_{s+i}v \text{ for some } i = 0, 1, \dots, k-1\} \tag{1.5}$$

has no $k+1$ coprimes.

The case $s = 1$, in which we have $\mathbb{N}_1(n) = [1, n]$, is of particular interest.

Conjecture 1:

$$f(n, k, 1) = |\mathbb{E}(n, k, 1)| \quad \text{for all} \quad n, k \in \mathbb{N}.$$

It seems that this conjecture of Erdös appeared for the first time in print in his paper [1] of 1962.

The papers [2] and [3] by Erdös, Sárközy and Szemerdi and the recent paper [7] by Erdös and Sarkozy are centered around this problem. Whereas it is easy to show that the conjecture is true for $k = 1$ and $k = 2$, it was proved for $k = 3$ by Szabo and Toth [6] only in 1985. Conjecture 1 can also be found in Section 3 of the survey [4] of 1973. In the survey [5] of 1980 one finds the