TOWARDS CHARACTERISING EQUALITY IN CORRELATION INEQUALITIES

Rudolf Ahlswede and Levon H. Khachatrian

Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
4800 Bielefeld 1
Germany

Institute of Problems
of Information and Automation
Armenian Academy of Sciences
Erevan–44 P. Sevak str. 1
Armenian
Visiting SFB 343

1. Introduction

For most of the basic inequalities in mathematics we know conditions, which completely specify the cases of equality. Many combinatorial correlation inequalities are special cases of the AD–inequality as is explained in the books [10], [11] or in [12].

However, for this inequality it seems to be difficult to classify the cases of equality. Certainly, this is even more difficult for the much more general inequalities of [12] and its relatives, which can be produced by the very same ideas of exploiting notions of expansiveness. In fact the *equality characterization problem* for these general inequalities constitutes by itself a rich area in combinatorial extremal theory. Closer to home their are the equality characterization problems for inequalities, which are *consequences* of the AD–inequality. Aharoni and Holzman ([9]) completely settled this for the Marica–Schönheim inequality. Another, though fairly special, still interesting case of AD could be handled by Beck [3].

It seems that the first study of this kind was made by Daykin, Kleitman, and West [5], who investigated the inequality

$$|A|\,|B| \le |L|\,|A \wedge B|, \tag{1.1}$$

where the lattice L is a product of finite chains and

$$A \wedge B = \{a \wedge b : a \in A, b \in B\}.$$

If L is a lattice of subsets of a finite set, then this inequality follows immediately from an inequality known to combinatorialists as Kleitman's inequality [3] and known to probabilists and physicists as Harris's inequality [19]. The more general inequality (1.1) was proved by Anderson [10] and by Greene/Kleitman [17].

Actually the product of chains is a distributive lattice and (1.1) extends to any distributive lattice, because as such it is a special case of FKG ([15]). This was noticed by Seymour and Welsh [14].

FKG in turn is a simple consequence of AD (see [12]). Our renewed interest in correlation inequalities came with our introduction and study of cloud–antichains ([1], [7]) and the connection to inequality (1.1), which we established in [8].

The main contributions of the present paper are two equality characterization results. They both continue and complete basic investigations of Daykin, Kleitman and West [5].

I. On pages 142–143 of [5] there is a detailed discussion about the difficulties in extending the results (Theorems 4, 5) basic for equality characterization in (1.1) for lattices, which are products of chains of equal length k, to lattices, which are products of chains of varying lengths, say $k_1, k_2, \dots, k_n$.

   We overcome these difficulties and also obtain the desired equality characterizations in Theorems 1, 2 (Section 3).