# OPTIMAL PAIRS OF INCOMPARABLE CLOUDS IN MULTISETS

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## 1. Introduction, basic results and problems

Let us be given the partially ordered set $\mathcal{P}_n([k]^n, \leq)$ , where $[k] = \{0, 1, 2, \ldots, k-1\}$ and $a = (a_1, a_2, \ldots, a_n) \leq b = (b_1, b_2, \ldots, b_n)$ iff $a_t \leq b_t$ for $t = 1, 2, \ldots, n$ . In the terminology of our earlier work ([1], [8], [9]) we call a pair $(A, B)$ with $A, B \subset [k]^n$ a cloud–antichain of length 2, if

$$a \not\leq b, a \not\geq b \ \text{ for all } \ a \in A, b \in B. \tag{1.1}$$

A short expression for (1.1) is: $A \supset\!\subset B$ .

We denote the set of these pairs by $\mathcal{CAC}(n)$ . The objects of our investigation are the functions $f_n : \{0, 1, \ldots, k^n\} \to \{0, 1, \ldots, k^n\}$ , $n \in \mathbb{N}$ , defined by

$$f_n(\alpha) = \max\big\{|B| : \exists (A, B) \in \mathcal{CAC}(n) \ \text{ with } \ |A| = \alpha\big\} \tag{1.2}$$

and a characterization of pairs $(A, B)$ which are optimal, that is, assume this bound. We denote by $\mathcal{O}(n)$ the set of all those optimal pairs.

In cases where we emphasize the dependence on parameter $k$ we also write $\mathcal{CAC}_k(n)$ , $\mathcal{O}_k(n)$ , $f_{n,k}(\alpha)$ , etc. instead of $\mathcal{CAC}(n)$ , $\mathcal{O}(n)$ , $f_n(\alpha)$ , etc.

Previous work is discussed in [9], where the best results prior to those in this paper can be found. They are all for the binary alphabet, i.e. $k = 2$ . Familiarity with this paper may be helpful but is not necessary for an understanding of the present results and proofs. We extend here first the key result of that paper to the case of general $k$ .

**Theorem 1.** *For every $\gamma, 0 \leq \gamma \leq k^{n-2}$ , a pair $(A, B) \in \mathcal{O}(n)$ with $|A| = \gamma$ exists, such that for some component all $a$ in $A$ have a $0$ and for some other component all $a$ in $A$ have a $k-1$ .*

From here we derive by an approach similar to (but not identical with) that of [9] the main recursion.

**Theorem 2.** *For every $\gamma, 0 \leq \gamma \leq k^{n-2}$ ,*

$$\begin{aligned} &\text{(i)} && f_n(\gamma) = (k-1)k^{n-1} + k f_{n-2}(\gamma) - (k-1)\gamma. \\ &\text{(ii)} && f_{n+2s}(\gamma) = (k^s - 1)(k^{n+s} - \gamma) + k^s f_n(\gamma) \ \textit{for } s \geq 0. \end{aligned} \tag{1.3}$$

The explicit characterization of all pairs $\big(\gamma, f_n(\gamma)\big)$ given in [9] does not seem to allow a reasonably simple extension to general $k$ . Therefore results concerning aspects of this characterization problem are already of interest.

Theorem 8 of [1] states that in the case $k = 2$ for $(A, B) \in \mathcal{CAC}(n)$

(a) $|A||B| \leq 2^{2n-4}$

(b) $\min\{|A|, |B|\} \leq 2^{n-2}$

and that these bounds are best possible.