DENSITY INEQUALITIES FOR SETS OF MULTIPLES

Rudolf Ahlswede and Levon H. Khachatrian

Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany

Abstract

For finite sets $A, B \subset \mathbb{N}$, the set of positive integers, consider the set of least common multiples $[A, B] = \big\{[a, b] : a \in A, b \in B\big\}$, the set of largest common divisors $(A, B) = \big\{(a, b) : a \in A, b \in B\big\}$, the set of products $A \times B = \{a \cdot b : a \in A, b \in B\}$, and the sets of their multiples $M(A) = A \times \mathbb{N}, M(B), M[A, B], M(A, B)$, and $M(A \times B)$, resp. Our discoveries are the inequalities

$$\mathbf{d}M(A, B)\mathbf{d}M[A, B] \geq \mathbf{d}M(A) \cdot \mathbf{d}M(B) \geq \mathbf{d}M(A \times B),$$

where $\mathbf{d}$ denotes the asymptotic density. The first inequality is by the factor $\mathbf{d}M(A, B)$ sharper than Behrend's well–known inequality. This in turn is a generalisation of an earlier inequality of Rohrbach and Heilbronn, which settled a conjecture of Hasse concerning an identity due to Dirichlet. Our second inequality does not seem to have predecessors.