ON GAUSSIAN MEASURE OF BALLS IN $\mathbb{H}$

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ABSTRACT. Different estimates are derived for the density function of a random variable $|Y-a|^2$, where Y is a Gaussian random element in a Hilbert space H with zero mean, depending on possible various values of a. The estimates imply upper bounds for probabilities $P(|Y-a|>r)$ with the same behaviour with respect to a and r as the asymptotically sharp results of Linde (1991) obtained for the case $r\to\infty$. Lower bounds for $(P(|Y-a|>r)$ are also constructed.

1. Introduction and Notation.

Let H be a real separable Hilbert space with an inner product $(\cdot,\cdot)$ and a norm $|x|=(x,x)^{1/2}$, $x\in H$. We denote by Y a H–valued Gaussian random element with zero mean and covariance operator V, i. e. the characteristic functional of Y has a form

$$E\exp\{i(x,Y)\}=\exp\{-(Vx,x)/2\},\quad x\in H,$$

where V is defined for all x,z from H by

$$(Vx,z)=E(Y,x)(Y,z).$$

Assume that the eigenvalues $\sigma_j^2, j=1,2,\dots$ of the covariance operator V satisfy

$$\sigma_1^2=\sigma_2^2=\dots=\sigma_k^2>\sigma_{k+1}^2\geq\sigma_{k+2}^2\geq\dots$$

for some $k\geq 1$ which is the multiplicity of σ_1^2 here and everywhere below. It is well known that there exists an orthonormal system $\{e_i\}_{i=1}^{\infty}$ in H such that $Ve_i=\sigma_i^2e_i$, or $Y=\sum_1^\infty Y_ie_i$, where $Y_i=(Y,e_i)$ are independent real Gaussian random variables with zero mean and variances σ_i^2. A density function of a random variable $|Y-a|^2$ will be denoted by $p(u,a)$, $u\geq 0$, $a\in H$.

Zolotarev (1961) proved the first sharp asymptotic result for $p(u,0)$

$$\lim_{u\to\infty}p(u,0)/f_k(u,0)=M(k),\tag{1}$$

where

$$M(k)=\prod_{j=k+1}^{\infty}(1-\sigma_j^2/\sigma_1^2)^{-1/2}$$

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and $f_k(u,0)$ is a density function of $\sum_1^k Y_i^2$, i. e.

$$f_k(u,0) = (2\sigma_1^2\Gamma(k/2))^{-1}(u/(2\sigma_1^2))^{k/2-1}\exp(-u/(2\sigma_1^2)),$$

where $\Gamma(\cdot)$ is gamma function. There are a lot of papers devoted to different generalizations of (1) on asymptotic behaviour of a Gaussian measure in a Hilbert space or in some Banach spaces (see e. g. Lifshits (1991) and Linde (1991) and bibliography in these papers). But in order to get generalizations of the law of the iterated logarithm to a Hilbert space (see e. g. Einmahl (1991)) or to study an accuracy of Gaussian approximation in infinite dimensional spaces (see e. g. survey by Bentkus et al. (1990)) one needs to know the behaviour of Gaussian measure of balls $B(a,r) = \{x \in H : |x-a| \le r\}$ not only for large but for all values of r.

One of the first results in this direction was obtained by Hoeffding (1964) who proved (cf. (1)) for all $u > 0$ and $k \ge 2$

$$p(u,0) \le M(k) f_k(u,0). \tag{2}$$

Later several upper estimates for $p(u,a)$ when $a \ne 0$ were derived in the papers on the central limit theorem in a Hilbert space (see the monographs by Paulauskas and Račkauskas (1989) and by Sazonov (1981) and the bibliography there and also Sazonov et al. (1988)). The discussion on these results see in Ulyanov (1991). However (2) does not follow from the above mentioned inequalities. In Ulyanov (1991) the inequality (2) was generalized in the following way: if $k \ge 2$ then for any $a \in H$

$$\begin{aligned} p(u,a) \le {} & M(k)(2\sigma_1^2\Gamma(k/2))^{-1} \\ & \times (u/(2\sigma_1^2))^{k/2-1}\exp(-(u^{1/2}-|a|)^2/(2\sigma_1^2)). \end{aligned} \tag{3}$$

Obviously, (3) coincides with (2) if $a = 0$, i. e. a equals the zero element in H. The Theorem in Richter and Ulyanov (1991) gave a hint that (3) can be improved under additional assumptions on a.

The aim of the present paper is to get these refinements of (3).

We give different upper bounds for $p(u,a)$ according to possible various values of a and k (see Theorem 1 and Remark after it).

In order to show how sharp the new results are, we construct upper and lower estimates for $P(|Y-a| > r)$ (see Theorem 2 and Corollary).

Our new estimates imply, in particular upper bounds for probabilities $P(|Y-a| > r)$ with the same behaviour with respect to a and r as the asymptotically sharp resutls of Linde (1991) obtained for the case $r \to \infty$ (cf. Theorem 3 and Remark 1 on p. 1274 in Linde (1991) and our Theorem 2).

2. Main Results.

Put for $a \in H$

$$\bar{a}_l = (a_1, \dots, a_l),$$

where $a_i = (a, e_i), i = 1, 2, \dots, l$.