VARIATION OF THE DISCRETE EIGENVALUES OF NORMAL OPERATORS

L. Elsner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany

S. Friedland
Department of Mathematics,
Statistics, and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60680
USA

Abstract. The Hoffman–Wielandt inequality which gives a bound for the distance between the spectra of two normal matrices, is generalized to normal operators A, B on a separable Hilbert space, such that $A - B$ is Hilbert–Schmidt.

1. Introduction

Let A, B be two n–by–n normal matrices. The famous Hoffman–Wielandt inequality states that the eigenvalues $\lambda_i,\ i = 1, \dots, n$ of A and $\mu_i,\ i = 1, \dots, n$ of B can be ordered in such a way that

$$\sum_{i=1}^{n} |\lambda_i - \mu_i|^2 \leq ||A - B||_F^2, \tag{1.1}$$

where

$$||C||_F^2 = \sum_{i,j=1}^{n} |c_{ij}|^2$$

is the Euclidean or Hilbert–Schmidt norm of a matrix $C = (c_{ij})$.

Several generalizations of this result have appeared in the literature.

Recently, Bhatia and Bhattacharya [B–B] obtained an analogous result for the joint eigenvalues of k–tuples of commuting normal matrices, see also Elsner [Els] for a simplified proof. In this note we will concentrate on the infinite dimensional case.

Let H be an infinite dimensional separable Hilbert space and $A : H \to H$ a bounded linear operator. We call a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of complex numbers an extended enumeration of the discrete spectrum of A if all discrete eigenvalues of A (i.e. isolated points of the spectrum with finite multiplicity as eigenvalues) belong to α counted with their multiplicity and all other elements of α are in the essential spectrum, i.e. they are limits of discrete eigenvalues, eigenvalues of infinite multiplicity or in the continuous spectrum. Let H be an infinite dimensional Hilbert space (not necessary separable) and $A : H \to H$ be a bounded normal operator. A is called diagonal if H has an orthonormal basis which consists entirely of eigenvectors of A.

Kato [Kat] has shown that for A, B hermitian there exist extended enumerations $\{\alpha_i\}, \{\beta_i\}$ of the discrete spectra of A, B such that

$$(1.2) \qquad \sum_{i=1}^{\infty} |\alpha_i - \beta_i|^2 \leq ||A - B||_F^2$$

provided that $A - B$ is a Hilbert–Schmidt operator. Here a compact operator $C : H \to H$ is Hilbert–Schmidt, if the singular values $s_j(C), \quad j = 1, 2, \dots$ satisfy

$$(1.3) \qquad ||C||_F^2 \equiv \sum_{j=1}^{\infty} s_j^2(C) < \infty.$$

Kato's result (1.2) was extended by Bhatia–Sinha [B–S] to unitary A, B. Bhatia–Elsner in [B–E] showed that (1.2) holds for A, B normal Hilbert–Schmidt operators. Here the essential spectrum is the point $z = 0$.

In this note we relax their assumptions by requiring A, B only to be normal with $A - B$ Hilbert–Schmidt.

Our main tool is Theorem 1. In the finite dimensional case it is a straightforward consequence of Birkhoff's theorem that the extreme points of the doubly stochastic matrices are the permutation matrices. In our case we have to use a selection procedure different from the one used in [B–E].

An immediate consequence is our first result (Theorem 2):
If A and B both are bounded diagonal operators then there exist extended enumerations $\{\alpha_i\}, \{\beta_i\}$ of the discrete spectra of A and B such that (1.2) holds.

Weyl in [Wey] proved that each bounded hermitian operator in a separable Hilbert space is a hermitian diagonal operator plus a compact one. J. von Neumann proved in [Neu] that the compact operator can be chosen to be Hilbert-Schmidt with an arbitrary small Hilbert-Schmidt norm. Later on Berg showed in [Ber] any normal operator A is a diagonal operator plus a compact one. If in addition the spectrum of A lies on a rectifiable curve then the compact operator can be chosen to be Hilbert-Schmidt with an arbitrary small Hilbert-Schmidt norm. Finally, Voiculescu in [Voi] proved that any bounded normal operator in a separable Hilbert space is a bounded diagonal operator plus a Hilbert-Schmidt operator with an arbitrary small Hilbert-Schmidt norm.