INFINITE PRODUCTS AND PARACONTRACTING MATRICES

L. Elsner

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131

33501 Bielefeld
FRG

1. Introduction.

In the investigation of chaotic iteration procedures for linear consistent systems matrices which are paracontracting with respect to some vector norm play an important role. It was shown in [EKN], that if $A_1, \ldots, A_m$ are finitely many k–by–k complex matrices which are paracontracting with respect to the same norm, then for any sequence $d_i,\ 1 \leq d_i \leq m,\ i = 1, 2, \ldots$ and any x_0 the sequence

$$x_{i+1} = A_{d_i} x_i \quad i = 1, 2, \ldots$$

is convergent. In particular $A^{(d)} = \lim\limits_{i \to \infty} A_{d_i} \ldots A_{d_1}$ exists for all sequences $\{d_i\}_{i=1}^{\infty} = d$. Hence those sets are examples of sets of matrices all infinite products of which converge. Such sets have been studied in [DL]. Following [DL], we call them LCP–sets.

In this note we investigate the question of necessity. As our main result we show that under the additional assumption that the mapping

$$d = \{d_i\}_{i=1}^{\infty} \to A^{(d)} = \lim_{i \to \infty} A_{d_i} A_{d_{i-1}} A_{d_1}$$

is continuous (which is equivalent to the set of fixed points of A_i being the same for all $1 \leq i \leq m$), a LCP–set is necessarily paracontracting with respect to some norm. In this sense paracontractedness is equivalent to the LCP–property.

2. Notations and known results.

Let $|| \; ||$ denote a vector norm in $\mathbb{C}^k$. A k –by– k matrix P is **paracontracting** with respect to $|| \; ||$, if for all x

$$Px \neq x \Leftrightarrow ||Px|| < ||x||.$$

We denote by $\mathcal{N}(|| \; ||)$ the set of all k –by– k matrices paracontracting w.r.t. $|| \; ||$.

For a bounded set $\Sigma = \Sigma_1$ of complex k –by– k matrices define $\Sigma_0 = \{I\}$ and for $n \geq 1$

$$\Sigma_n = \{M_1 \ M_2 \dots M_n : M_i \in \Sigma\},$$

the set of all products of matrices in Σ of length n . Let $\Sigma = \{A_1, \dots , A_m\}$ be finite. For $d = (d_1, d_2, \dots) \in \{1, \dots , m\}^{\mathbb{N}}$, i.e. $1 \leq d_i \leq m$ for $i \in \mathbb{N}$ define

$$A^{(d)} = \lim_{n \to \infty} A_{d_n} A_{d_{n-1}} \dots A_{d_1},$$

if the limit exists.
Σ is an LCP–set (left–convergent–product), if for all $d \in \{1, \dots , m\}^{\mathbb{N}}$ the limit $A^{(d)}$ exists. The function $d \to A^{(d)}$ mapping $\{1, \dots , m\}^{\mathbb{N}}$ into the space of k–by–k matrix is called the **limit function**.

We note in passing that in [DL] also the right–convergent–product property (RCP) was introduced. For convenience we restrict our considerations to the left convergence case.

Introducing in $\{1, \dots , m\}^{\mathbb{N}}$ the metric

$$\text{dist } (d, d') = m^{-r} \qquad r \text{ smallest index such that } d_r \neq d'_r$$

we define the concept of a **continuous limit function** in the standard way.

Σ is **product bounded**, if there exists $\Delta > 0$ such that

$$||A|| \leq \Delta \text{ for all } A \in \Sigma_n, \ n = 1, 2, \dots$$

Here $|| \ ||$ denotes any matrix norm. Obviously this concept is independent of the norm. G. Schechtman has proved that LCP–sets are product bounded (see Theorem I of [BW]). We have the following statement.

Lemma 1. *For a set* Σ *of* k *–by–* k *matrices the following are equivalent:*

(1) Σ *is product bounded.*
(2) $\exists$ *vector norm* $|| \ ||$ *such that* $||Ax|| \leq ||x||$ *for all* $A \in \Sigma, \ x \in \mathbb{C}^k$.
(3) $\exists$ *multiplicative matrix norm* $|| \ ||$ *such that* $||A|| \leq 1$ *for all* $A \in \Sigma$.

Proof.
As (2) $\implies$ (3) (the operator norm is multiplicative) and (3) $\implies$ (1) are obvious, only (1) $\implies$ (2) has to be shown.
For some vector norm ν define the norm

$$||x|| = \sup_{n \geq 0} \{ \sup_{A \in \Sigma_n} \nu(Ax)\}$$

which is finite by (1). Then $||Ax|| \leq ||x||$ for all $A \in \Sigma$. ■

Remark: This result could also be derived from [RS].

For a given matrix norm $|| \ ||$ and bounded Σ let

$$\widehat{\rho}_n = \widehat{\rho}_n(\Sigma) = \max\{||A||, A \in \Sigma_n\}$$