Number theoretic correlation inequalities for Dirichlet densities

Rudolf Ahlswede and Levon H. Khachatrian

Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany

Abstract

Our main discovery is the inequality

$$\underline{\mathbf{D}}\,(A,B)\,\underline{\mathbf{D}}[A,B] \;\geq\; \underline{\mathbf{D}}\,A\;\underline{\mathbf{D}}\,B,$$

where A, B are arbitrary sets of positive integers, $(A,B) = \big\{(a,b) : a \in A, b \in B\big\}$ is the set of largest common divisors, $[A,B] = \big\{[a,b] : a \in A, b \in B\big\}$ is the set of least common multiples, and $\underline{\mathbf{D}}$ denotes the lower Dirichlet density. It is much more general than our recent inequality for multiples of sets, which in turn is sharper than Behrend's well–known inequality.

We also extend another recently discovered inequality, which does not seem to have number theoretic predecessors.

1 Introduction

The starting point of our investigations are two new inequalities of [3], which concern asymptotic densities of sets of multiples of certain sets of positive integers. They are readily stated.

For sets $A, B \subset \mathbb{N}$, the set of positive integers, consider the set of least common multiples $[A, B] = \big\{[a, b] : a \in A, b \in B\big\}$, the set of largest common divisors $(A, B) = \big\{(a, b) : a \in A, b \in B\big\}$, the set of products $A \times B = \{a \cdot b : a \in A, b \in B\}$, and the sets of their multiples $M(A) = A \times \mathbb{N}, M(B), M[A, B]$, $M(A, B)$, and $M(A \times B)$, resp.

We use the abbreviations $\langle 1, n\rangle = \{1, 2, \dots, n\}$ and for any set $C \subset \mathbb{N}$ $C_n = C \cap \langle 1, n\rangle$.

The asymptotic density $\mathbf{d}C$ is defined as $\mathbf{d}C = \lim\limits_{n\to\infty} C_n/n$, if the limit exists. The lower (resp. upper) asymptotic densities $\underline{\mathbf{d}}C$ and $\overline{\mathbf{d}}C$ are defined similarly with lim replaced by lim inf (resp. lim sup).

The discoveries in [3] are the inequalities

$$\mathbf{d}M(A, B)\mathbf{d}M[A, B] \geq \mathbf{d}M(A) \cdot \mathbf{d}M(B) \tag{1.1}$$

and

$$\mathbf{d}M(A) \cdot \mathbf{d}M(B) \geq \mathbf{d}M(A \times B), \tag{1.2}$$

where A and B are **finite**.

The first inequality is by the factor $\mathbf{d}M(A, B)$ sharper than Behrend's well–known inequality ([9], [25], [16]). This in turn is a generalisation of an earlier inequality of Rohrbach [27] and Heilbronn [18], which settled a conjecture of Hasse concerning an identity due to Dirichlet (see [27]). A simple proof was given in [28] via a probabilistic inequality, which actually is a special case of the earlier FKG–inequality [15].

The second inequality does not seem to have predecessors in number theory.

We mentioned already in [4] that structural similarities between Behrend's inequality and the AD–inequality (see Section 2) led us to conjecture the first inequality.

Now we go further in three directions. We mention first that these similarities are more than just analogies. In fact, we show in Section 2 that (1.1) is a *consequence* of the AD–inequality. Thus we have a unified view and the AD–inequality now not only includes several correlation inequalities from statistical physics ([15]) resp. probability theory ([17], [19]) and combinatorics ([22], [29], [12], [7]), but also well–known density inequalities ([27], [18], [9]) in number theory.

Secondly, this approach gives more than just another proof of (1.1), because it works for *arbitrary subsets* $C, D \subset \mathbb{N}$ and not just sets of multiples $M(A)$, $M(B)$ with A, B finite (Theorem 1 in Section 2).