Generalized one-point extensions

Peter Dräxler

Introduction

Let A be a finite dimensional algebra over an algebraically closed field k. Suppose that there is an indecomposable projective left A-module P satisfying $\dim_k \mathrm{End}_A(P) = 1$. We recall from [Dr2] that the full subcategory $K(A,P)$ of the category A-mod of all finitely generated left A-modules is just given by all modules $V \in A$-mod satisfying $\mathrm{Ext}^1_A(V, \mathrm{fac}\, P) = 0$. Of particular interest is the full subcategory $K_P(A,P)$ of $K(A,P)$ which consists only of those $V \in K(A,P)$ where $\mathrm{Hom}_A(P,X) \neq 0$ for each direct summand X of V.

The importance of $K(A,P)$ comes from the fact that the fiber sum functor F_P introduced in [Dr1] furnishes a full and dense functor from the subspace category $\check{\mathcal{U}}(K(A,P), \mathrm{Hom}_A(P,-))$ to the category A-mod. This shows immediately that A is of finite representation type if and only if the vectorspace category $(K(A,P), \mathrm{Hom}_A(P,-))$ is of finite subspace type. Concerning tame representation type the situation was clarified in [Dr2]. Namely, if we write P as Ae for some primitive idempotent e, then the algebra A is of tame representation type if and only if the factor algebra A/AeA is of tame representation type, the vectorspace category $(K_P(A,P), \mathrm{Hom}_A(P,-))$ is of tame subspace type and for every $n \in \mathbb{N}$ there are up to isomorphism only finitely many indecomposable modules $X \in K_P(A,P)$ such that $\dim_k X \leq n$.

It is a well-known phenomenon in representation theory of algebras that finite or tame type is much easier to verify and to study for vectorspace categories rather than for algebras. Therefore one looks for reductions from module categories to subspace categories. [G-V], [Ri1] ,[Ri2] is a certainly very incomplete list of references where reductions of this kind were applied for tame type. The above characterization of tameness of an algebra A via the vectorspace category $(K(A,P), \mathrm{Hom}_A(P,-))$ should give another efficient reduction tool for the presently very active reasearch on tame algebras, once this vectorspace category can actually be calculated. It is the major aim of this paper to investigate the structure of this vectorspace category for a big class of algebras A and modules P.

To sketch our results, we recall from [Dr2] that the well-known one-point extension resp. coextension technique applied e.g. in [Ri1], [Ri2] can be considered as a special case of the fiber sum functor method if one chooses for P a simple projective module. In this case the algebra A has the form $A = \begin{pmatrix} B & 0 \\ R & k \end{pmatrix}$ where R lies in the category mod-B of finitely generated right B-modules. The point is that the vectorspace category $(K(A,P), \mathrm{Hom}_A(P,-))$ can almost be identified with $(B\text{-mod}, R \otimes_B -)$. More precisely one has to pass from $K(A,P)$ to the full subcategory $K(A,P) \setminus \mathrm{add}\, P$ of objects admitting no summand isomorphic to P to arrive at isomorphic vectorspace categories. This is of course no essential loss of information.

As we want to deal with non-simple P, we need some more notations. Let us assume that A is basic. Hence we can write A as $k[\Delta]/I$ where Δ is a finite quiver and I is an admissible ideal of the path algebra $k[\Delta]$ of the quiver Δ (see [Ga]). Denoting by $e(x)$ the primitive idempotents of A induced by the vertices x of Δ, we obtain $P = Ae(s)$ for some $s \in \Delta_0$. We remark that we compose arrows from right to left.

It is convenient to introduce a transitive and reflexive relation $\leq_A$ on the vertex set Δ_0 of Δ by defining $y \leq_A x$ provided there is a path in Δ from x to y. Of course this relation will usually not be antisymmetric, but we require that the interval $[s,s]_A$ with respect to this relation consists only of the element s. Defining e^s resp. e_s as the sum of all $e(x)$ such that $x \not\geq_A s$ resp. $x \not\leq_A s$, we construct the algebras $A^s := e^s A e^s$ resp. $A_s := e_s A e_s$ which can be considered as factor algebras of $A/s := A/Ae(s)A$ in the natural way. We also need the idempotent $\bar{e}^s = e^s + e(s)$ and the corresponding algebra $\bar{A}^s = \bar{e}^s A \bar{e}^s$ which is not a factor of A/s but still of A. Observe that the category $\bar{A}^s$-proj of projectives in $\bar{A}^s$-mod can be considered as a full subcategory of A-proj.

The vertex s is called decomposing if every indecomposable module in A/s-mod lies either in A^s-mod or in A_s-mod. In this case the category $K(A,P)$ consists of the three parts $\bar{A}^s$-proj, K^+ and K^- such that there are $R^+ \in A^s$-mod and $R^- \in$ mod-A_s with the property that (up to some technicalities which we will discuss in section 2) the vectorspace categories $(K^+, (\mathrm{Hom}_A(P,-))$ resp. $(K^-, (\mathrm{Hom}_A(P,-))$ can be identified with $(A^s\text{-mod}, \mathrm{Hom}_{A^s}(R^+,-))$ resp. $(A_s\text{-mod}, R^- \otimes_{A_s} -)$. Hence we are in a situation similar pleasent as one-point coextension.

Actually if P is simple projective or equivalently s is a sink of Δ, then s is obviously decomposing. Moreover in this case $\bar{A}^s$-proj collapses to $\mathrm{add}\, P$ and K^+ is superfluous. This shows again that we are just generalizing the one-point coextension situation. As it turns out in section 3 that conversely a decomposition of $K(A,P)$ into three parts as above forces s to be decomposing, we call algebras posessing a decomposing vertex s also generalized one-point extensions by s.

Of course in applications the question arises how the parts K^+ and K^- are related by homomorphisms. This problem is simplified by the observarion that one may path from the category $K_P(A,P)$ to the factor category $K_P(A,P)/I_P$ where I_P denotes the kernel of the functor $\mathrm{Hom}_A(P,-)$ without changing the representation type. We show in section 2 that there are no non-zero maps from K^-/I_P to K^+/I_P and discuss the maps in the opposite direction in section 4 by investigating the quiver of $K_P(A,P)/I_P$.

In section 5 we will finish the construction of the quiver of $K_P(A,P)/I_P$ under certain finiteness and directedness conditions. In particular as an application of our access we are able to complete the description of hammocks of representation directed algebras presented in [Sc].

Section 6 is devoted to examples where our methods make it possible to understand the whole structure of $(K_P(A,P)/I_P, \mathrm{Hom}_A(P,-))$.

We finish the introduction by pointing out that until section 3 all our results are valid also for algebras over arbitrary fields or even artin algebras provided instead of requiring $\dim_k \mathrm{End}_A(P) = 1$ one demands that $\mathrm{End}_A(P)$ is a skew field H and consequently replaces k by H at the appropriate places. In section 4 of course more care would have to be taken