Menger's theorem for a countable source set

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Dedicated to Paul Erdős
on the occasion of his 80th birthday

Paul Erdős has conjectured that Menger's theorem extends to infinite graphs in the following way: whenever A, B are two sets of vertices in an infinite graph, there exist a set of disjoint A–B paths and an A–B separator in this graph so that the separator consists of a choice of precisely one vertex from each of the paths. We prove this conjecture for graphs that contain a set of disjoint paths to B from all but countably many vertices of A. In particular, the conjecture is true when A is countable.

Introduction

If there is any conjecture in infinite graph theory whose fame has clearly transcended the boundaries of the field, it is the following infinite version of Menger's theorem, conjectured by Erdős:

Conjecture. (Erdős)
Whenever A, B are two sets of vertices in a graph G, there exist a set of disjoint A–B paths and an A–B separator in G so that the separator consists of a choice of precisely one vertex from each of the paths.

Here, G may be either directed or undirected, and either finite or infinite. If G is finite, the statement is clearly a reformulation of Menger's theorem. A set of A–B paths together with an A–B separator as above will be called an *orthogonal paths/separator pair*. *orthogonal pair*

Although Erdős's conjecture has been proved for countable graphs [2], a full proof still appears to be out of reach. However, no other conjecture in infinite graph theory has inspired as interesting a variety of partial or related results as this one; see [4] for a survey and list of references.

The main aim of this paper is to prove a lemma which, in addition to implying the results stated in the abstract, might play a role in an overall proof of the conjecture by induction on the size of G. Briefly, the lemma implies that if the conjecture is true for all graphs of size κ, where κ is any infinite cardinal, then it is true also for arbitrary graphs provided the source set A is no larger than κ. (In particular, we see that the conjecture holds for any graph if A is countable.) Now if $|A| = |G| = \lambda$ and the conjecture holds for all graphs of size $< \lambda$, the lemma enables us to apply the induction hypothesis to G with A replaced by its smaller subsets A'; we may then try to combine the orthogonal paths/separator pairs obtained between these A' and B to one between A and B. We must point out, however, that such a proof of Erdős's conjecture will by no means be straightforward, and it is not the only possible approach.

1. Definitions and statement of the main result

All the graphs we consider will be directed; undirected versions of our results can be recovered in the usual way by replacing each undirected edge with two directed edges pointing in opposite directions. An edge from a vertex x to a vertex y will be denoted by xy. When G is a graph, then $\overleftarrow{G}$ denotes the graph obtained from G by reversing all its edges.

xy $\overleftarrow{G}$

Paths, likewise, will be directed, and we usually refer to them by their vertex sequence. If $P = x \dots y$ is a path and v, w are vertices on P in this order, then vPw denotes the subpath of P from v to w. Similarly, we write Pv and vP for initial and final segments of P, $P\mathring{v}$ for $Pv - v$, $\mathring{v}P$ for $vP - v$, and so on. If $Q = y \dots z$ is another path, and $P \cap Q = \{ y \}$, then PyQ denotes the path obtained by concatenating P and Q.

$\mathring{v}$

Let X, Y be sets of vertices in a graph. An X–Y *path* is a path from X to Y whose inner vertices are neither in X nor in Y. If x is a vertex, then a set of $\{ x \}$–Y paths that are disjoint except in x is an x–Y *fan*; the fan is *onto* if every vertex in Y is hit. Similarly, a set of X–y paths that are disjoint except in y is an X–y *fan*.

X–Y path, *fan*

A *warp* is a set of disjoint paths. When $\mathcal{W}$ is a warp, we write $V[\mathcal{W}]$ for the set of vertices of the paths in $\mathcal{W}$, and $E[\mathcal{W}]$ for the set of their edges. Similarly, we write *in* $[\mathcal{W}]$ for the set of initial vertices of the paths in $\mathcal{W}$, and *ter* $[\mathcal{W}]$ for the set of their terminal vertices. For a vertex $x \in V[\mathcal{W}]$, we denote the path in $\mathcal{W}$ containing x by $Q_{\mathcal{W}}(x)$, or briefly $Q(x)$. For $x \notin V[\mathcal{W}]$, we put $Q_{\mathcal{W}}(x) := \{ x \}$. A warp consisting of A–B paths is an A–B *warp*. By $\overleftarrow{\mathcal{W}}$ we denote the warp in $\overleftarrow{G}$ consisting of the reversed paths from $\mathcal{W}$.*

warp, *in* $[\mathcal{W}]$, *ter* $[\mathcal{W}]$, $Q_{\mathcal{W}}(x)$, $\overleftarrow{\mathcal{W}}$

Let $G = (V, E)$ be a graph and $A, B \subseteq V$. A set $S \subseteq V$ *separates* A from B in G if every path in G from A to B meets S. Note that A and B may intersect, in which case clearly $A \cap B \subseteq S$.

A triple $\Gamma = (G, A, B)$ as above will be called a *web*. The web $(\overleftarrow{G}, B, A)$ is denoted by $\overleftarrow{\Gamma}$. An A–B warp $\mathcal{W}$ with *in* $[\mathcal{W}] = A$ is a *linkage* in Γ, and Γ

web, $\overleftarrow{\Gamma}$

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* Clearly, $\overleftarrow{\overleftarrow{\mathcal{W}}} = \mathcal{W}$. We shall use this fact as an excuse to denote warps in $\overleftarrow{G}$, if they are introduced afresh rather than being obtained from a warp in G, by $\overleftarrow{\mathcal{W}}$ etc. straight away; their reversals in G will then be denoted by $\mathcal{W}$. The idea here is to avoid the counter-intuitive practice of having a warp $\mathcal{W}$ in $\overleftarrow{G}$ and a resulting warp $\overleftarrow{\mathcal{W}}$ in G. This convention, if not its explanation, should help the reader to avoid any warps in his or her intuition when such things have to be discussed briefly in Section 4.