Singular values and doubly stochastic matrices

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Abstract The Hadamard square of any square matrix A is bounded above and below by some doubly stochastic matrices times the square of the largest and the smallest singular values of A.

§1. Introduction

For a set $D \subset \mathbf{C}$ let $M_n(D)$ be the set of $n \times n$ complex valued matrices with entries in D. Denote by $\Omega_n \subset M_n(\mathbf{R}_+)$ the convex set of doubly stochastic matrices. Let $A = (a_{ij})_1^n \in M_n(\mathbf{C})$. Set $A^{\circ 2} = A \circ \bar{A} = (a_{ij}\bar{a}_{ij})_1^n \in M_n(\mathbf{R}_+)$. Denote by $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$ the singular values of A. If $\sigma_1(A) = ... = \sigma_n(A) = s > 0$, then A is an s-multiple of a unitary matrix and $A^{\circ 2}$ is an s^2-multiple of a doubly stochastic matrix. In this paper we generalize the above fact to arbitrary complex valued square matrices.

Main Theorem. *Let $A \in M_n(\mathbf{C})$. Then there exist $A_1, A_2 \in \Omega_n$ so that*

$$\sigma_n(A)^2 A_1 \leq A^{\circ 2} \leq \sigma_1(A)^2 A_2.$$

Here the inequalities are taken in the entrywise sense. The lower estimate for $A^{\circ 2}$ seems to be much deeper than the upper estimate. We bring applications to graph theory, permanents and to perturbation estimates on the eigenvalues of matrices.

§1. Proof of the Main Theorem

Let $B = (b_{ij})_1^n \in M_n(\mathbf{R}_+)$. For $I, J \subset \{1, ..., n\}$ let

$$B_{IJ} = \sum_{i \in I, j \in J} b_{ij}.$$

Denote by $|I|, |J|$ the cardinalities of I, J respectively. B - is called $(0 \leq) t$ doubly super stochastic (d.sp.s.) if $B \geq tB_1$ for some $B_1 \in \Omega_n$. In [**Cru**] Cruse showed that B is t d.sp.s. iff

$$B_{IJ} \geq t(|I| + |J| - n), \forall\, I, J \subset \{1, ..., n\}.$$

Set

$$\mu(B) = \min_{I, J \subset \{1,...,n\}, |I|+|J|>n} \frac{B_{IJ}}{|I| + |J| - n}.$$

Cruse's theorem yields that B is $\mu(B)$ d.sp.s.. The first part of the Main Theorem follows from the following lemma.

(1.1) Lemma *Let $A \in M_n(\mathbf{C})$. Then*

$$\mu(A^{\circ 2}) \geq \sigma_n^2(A).$$

Proof. It is enough to assume that A is nonsingular. Let $P, Q \in \Omega_n$ be permutation matrices. Clearly, PAQ has the same singular values as A and $\mu((PAQ)^{\circ 2}) = \mu(A^{\circ 2})$. Thus, it suffices to show that

$$\sum_{i=1,j=1}^{i=p,j=q} |a_{ij}|^2 \geq (p + q - n)\sigma_n^2(A), 1 \leq p \leq q, n < p + q.$$

Let

$$D_p = diag(d_1, ..., d_n), d_1 = ... = d_p = 1, d_{p+1} = ... = d_n = 0.$$

Set $A' = D_p A D_q$. Let

$$V = \{x : x \in \mathbf{C}^n, (I - D_q)x = (I - D_p)Ax = 0\}.$$

It then follows that $m = dim(V) \geq p + q - n$. Moreover, $(A - A')V = 0$. Set $W = AV = A'V$. We can view A and A' as representing the same linear transformation $T : V \to W$. Furthermore, V, W are inner product spaces induced by the standard inner product in $\mathbf{C}^n$. Observe next

$$\sigma_m(T) = \min_{x \in V \setminus \{0\}} \frac{|Ax|}{|x|} \geq \min_{x \in \mathbf{C}^n \setminus \{0\}} \frac{|Ax|}{|x|} = \sigma_n(A).$$