Minuscule Representations

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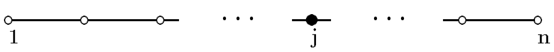
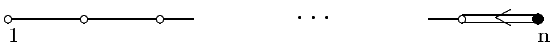
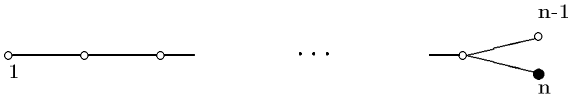
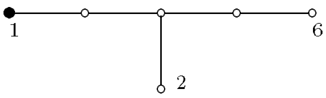
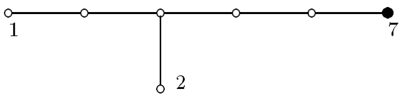
1 Introduction

Let G be a semisimple universal Chevalley group over an algebraically closed field F of arbitrary characteristic, W be its Weyl group, Ψ its root system and Π a fundamental set of roots in Ψ. In this paper we are concerned with the irreducible FG-modules corresponding to the *minuscule* fundamental weights (see Section 2 and note we include the trivial weight $\lambda = 0$). Such FG-modules are called minuscule. More particularly, we are interested in the restriction of minuscule modules to parabolic subgroups of G. It is well-known, see [6, Ch. VIII, §7.3], that when Ψ is indecomposable the non-trivial minuscule weights correspond to the following fundamental modules:

Type	Dynkin Diagram	Usual Name	Dimension
$A(n,j)$		j-th exterior power of natural module	$\binom{n+1}{j}$
$B(n,n)$		spin	2^n

$C(n,1)$		natural	$2n$
$D(n,1)$		natural	$2n$
$D(n,n)$		1/2-spin	2^{n-1}
$E(6,1)$		minimal	27
$E(7,7)$		minimal	56

Here and elsewhere we denote by $A(n,j)$, $B(n,n)$, etc. the highest weight module $V(\lambda_j)$ for A_n and $V(\lambda_n)$ for B_n, etc.. The cases $D(n,n-1)$ and $E(6,6)$ are not listed, as they are, respectively, algebraically conjugate to $D(n,n)$ and dual to $E(6,1)$.

When λ is a minuscule weight, [6] tells us that the weight spaces of $V(\lambda)$ form a single orbit under the action of W. That is the weight spaces are in one-to-one correspondence with the cosets W/W_J where W_J is the stabilizer in W of λ. In [13] Proctor exploits this correspondence. He proves [13, Proposition 3.1] that if an irreducible *Bruhat poset* is a *lattice*, then either W is of type G_2 or the poset is isomorphic to the poset induced on the W-orbit of a minuscule weight with respect to the usual ordering on weights. In another direction minuscule modules arise in [11] when char$F$ is odd, as the only modules (up to Frobenius twists and embeddings of the field into an algebraically closed field) on which both classes of root groups act quadratically. In a further direction, Hiller [9], is able to apply results about minuscule posets to obtain information about the *Schubert calculus* of G/P, where P is the stabilizer in G of a maximal weight space in a minuscule module.

Assume that W_J is the stabilizer in W of a minuscule weight λ. A first step in understanding restrictions of a minuscule module to parabolic subgroups in terms of the geometric structure of W/W_J is made in Theorem 2.4. This theorem tells us that the minuscule FG-module $V = V(\lambda)$ restricted to the semisimple part of a Levi complement, L'_K, behaves in the best possible way: it splits into a direct sum of minuscule FL'_K-modules. Moreover, Theorem 2.4 says that the direct summands are determined by the Ψ_K-dominant weights $w\lambda$ where w is from the set of minimal length (W_K, W_J) double coset representatives in W. So, in particular, there is a one-to-one relationship between (W_K, W_J) double cosets in W and L'_K-summands in $V|_{L'_K}$, the restriction of V to L'_K. Theorem 2.8 asserts that this situation is exactly the same for all fields. A partial converse of Theorem 2.4 is proved in Proposition 2.11.