THE GENUS OF THE BARNES-WALL LATTICE

Rudolf Scharlau Boris B. Venkov

1 Introduction

In this paper, we combine the general concept of the root system of a Euclidean lattice as studied in [B-S] with the "main lemma" Proposition 1 of [Ve] to obtain the full classification of all 2-elementary totally even 16-dimensional lattices of determinant 2^8. It turns out that this genus of lattices consists of 24 isometry classes. Like in the case of 24-dimensional, even unimodular lattices, 23 of them are **reflective** in the sense that the root system has maximal rank 16. We derive a list of 'possible' root systems, which are subject to two essential restrictions. It turns out that for each root system in our list, there exists a unique lattice.

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2 Results

By 'Euclidean lattice', or 'lattice' for short, we mean as usual a $\mathbb{Z}$-lattice of full rank in a rational vector space with a positive definite scalar product (x, y). The lattice is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$, and ***p*-elementary**, for some prime number p, if $pL^\# \subseteq L$. Here $L^\# = \{y \in \mathbb{Q}L \mid (x, y) \in \mathbb{Z} \text{ for all } x \in L\}$ denotes the dual lattice of L.

By a left upper index $\alpha \in \mathbb{Q}^+$, we denote scaling: ${}^\alpha L$ equals L as a lattice, but the form is multiplied by α. If L is p-elementary, then ${}^p(L^\#)$ is again (integral and) p-elementary. In this paper, we specialize to the case $p = 2$, and we write ${}^2(L^\#) = L^\natural$ for short. We say that L is **totally even** if both L and $L^\natural$ are even, i.e. $(x, x) \in 2\mathbb{Z}$ for all $x \in L$ and $(y, y) \in \mathbb{Z}$ for all $y \in L^\#$.

For a given dimension n and determinant $\det L = |L^{\#}/L| = 2^r$, all 2-elementary totally even lattices form one genus. They exist if and only if either $r \in \{0, n\}$ and $n \equiv 0(8)$, or $0 < r < n$, $r \equiv 0(2)$, $n \equiv 0(4)$. If $n = 4m$ and $r = 2m$, the mapping $L \to L^{\natural}$ is an involution on (the isometry classes of) this genus. A $\natural$-invariant representative is $mD_4 = D_4 \perp .... \perp D_4$, where D_4 as usual denotes the D_4-root lattice. Another, prominent, lattice in the genus of $4D_4$ is the Barnes-Wall lattice $BW = BW_{16}$ which has no vector of norm 2 and again satisfies $BW \cong BW^{\natural}$.

From [B-S] we recall the definition

$$\mathsf{R}(L) := \{v \in L \mid v\,\text{primitive},\ 2(v, L)/(v, v) \subseteq \mathbb{Z}\}$$

of the **root system** of an arbitrary lattice L. The quadratic form is part of the structure of such root systems R. Therefore, scaling ${}^{\alpha}\mathsf{R}$ makes sense, and the irreducible ones are ${}^{\alpha}\mathsf{A}_l, {}^{\beta}\mathsf{B}_l, {}^{\gamma}\mathsf{C}_l, \ldots$. The lattice generated by a root system A, B, C, $\ldots$, R is denoted by $A, B, C, \ldots, R$. Specializing again to 2-elementary lattices, we observe, that

$$\mathsf{R}(L) = \{v \in L \mid (v, v) = 2\} \ \dot{\cup}\ \{v \in 2L^{\#} \mid (v, v) = 4\} =: \mathsf{R}_{sh} \dot{\cup}\ \mathsf{R}_{lo}$$

('short' and 'long' roots). Thus, the possible irreducible components of $\mathsf{R}(L)$ are A_l, ${}^2\mathsf{A}_l$, ${}^2\mathsf{B}_l$, C_l, D_l, ${}^2\mathsf{D}_l$, F_4, E_6, ${}^2\mathsf{E}_6$, E_7, ${}^2\mathsf{E}_7$, E_8, ${}^2\mathsf{E}_8$.

By "Niemeier lattice" we mean a 24-dimensional even unimodular lattice. A full classification of this genus of lattices into 24 isometry classes has been given by Niemeier in [Nie], and was simplified in [Ve]. The purpose of this note is to annouce and partially prove a classification result for the genus of $4D_4$ which is completely analogous to the classification of Niemeier lattices in the form presented in [Ve]. Incidently there are, again, precisely 24 isometry classes. In the following, h denotes the Coxeter number of an irreducible root system R. It is characterized by the following formula:

$$(1) \qquad \sum_{v \in \mathsf{R}} \frac{(v, x)^2}{(v, v)} = h \cdot (x, x) \text{ for all } x \in \mathbb{Q}\mathsf{R}.$$

See e.g. [Bou], Chap.V, §6.2, p. 121. By $\mathcal{G}$ we denote the genus consisting of all integral, 2-elementary, totally even lattices of dimension 16 and determinant 2^8.

Theorem 1 *Let L be a lattice in $\mathcal{G}$ and $\mathsf{R} = \mathsf{R}(L)$ its root system. Then the following properties hold:*

a) $\mathsf{R} = \emptyset$ *or* rank $\mathsf{R} = 16$.

b) If $\mathsf{R} \neq \emptyset$, *then*

(i) the Coxeter number h is the same for all irreducible components of R,

(ii) $|\mathsf{R}_{sh}| = |\mathsf{R}_{lo}|$,