On the Four Terms in the Middle Theorem for almost split sequences

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Recently Liu [L] has proved the following result, which generalizes remarkably the wellknown Four Terms in the Middle Theorem of Bautista and Brenner [BB].

Liu's Theorem *Let Λ be an artin algebra and let $0 \to X \to \coprod_{i=1}^{r} Y_i \to Z \to 0$ be an almost split sequence in the category of finitely generated Λ-modules such that all Y_i's are indecomposable. Suppose that X has a projective predecessor and Z has an injective successor in the Auslander–Reiten quiver Γ_Λ of Λ. Then $r \le 4$ and $r = 4$ implies that one of the Y_i's is projective-injective and the others are neither projective nor injective.*

We will show that Liu's proof can be translated into a purely combinatorial one. Before we state our combinatorial result which implies the above-mentioned theorem, let us fix some terminology.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a *quiver*, that is, a locally finite oriented graph with set of *vertices* Γ_0 and set of *arrows* Γ_1. Suppose that Γ contains neither loops nor multiple arrows. Given a vertex x, denote by x^+ the set of vertices y such that there is an arrow $x \to y$; the set x^- consists of all vertices y such that there is an arrow $y \to x$. A pair (Γ, τ) is called a *translation quiver*, if $\tau\colon \Gamma_0' \to \Gamma_0$ is an injective map for some subset $\Gamma_0' \subseteq \Gamma_0$, satisfying $(\tau x)^+ = x^-$ for all $x \in \Gamma_0'$. It is convenient to put $\tau^0 x = x$ for all $x \in \Gamma_0$. The vertices in $\Gamma_0 \setminus \Gamma_0'$ are called *projective*; those in $\Gamma_0 \setminus \tau\Gamma_0'$ are called *injective*. Let $\delta\colon \Gamma_1 \to \mathbb{N} \times \mathbb{N}$ be a map and denote the values by $\delta(\alpha) = (\delta_{x,y}, \delta'_{x,y})$ for each arrow $\alpha\colon x \to y$. The triple (Γ, τ, δ) is called a *valued translation quiver*, if the following conditions are satisfied for all non-projective vertices x.

($\delta 1$) $\quad \delta'_{\tau x, y} = \delta_{y,x}$ for all $y \in x^-$.

($\delta 2$) $\quad \delta_{\tau x, y} = \delta'_{y,x}$ for all $y \in x^-$.

Finally, a map $\ell\colon \Gamma_0 \to \mathbb{N}$ is called an *additive length function* for (Γ, τ, δ), if the following conditions are satified for all vertices x.

($\ell 1$) $\quad \ell(x) + \ell(\tau x) = \sum_{y \in x^-} \delta_{y,x} \ell(y)$, if x is non-projective.

($\ell 2$) $\quad \ell(x) > \sum_{y \in x^-} \delta_{y,x} \ell(y)$, if x is projective.

($\ell 3$) $\quad \ell(x) > \sum_{y \in x^+} \delta'_{y,x} \ell(y)$, if x is injective.

The following is the main result of this note.

Theorem *Let (Γ, τ, δ) be a valued translation quiver and ℓ be an additive length function. Let x be a vertex having a projective predecessor and an injective successor. Then $\sum_{y \in x^+} \delta'_{x,y} \leq 4$ and equality implies that x is non-injective and that x^+ contains a projective vertex.*

The following consequence is the combinatorial version of Liu's theorem.

Corollary *Let (Γ, τ, δ) be a valued translation quiver and ℓ be an additive length function. Let x be a non-injective vertex such that x has a projective predecessor and $\tau^{-1}x$ has an injective successor. Then $\sum_{y \in x^+} \delta'_{x,y} \leq 4$ and equality implies that x^+ contains a projective-injective vertex y_0 with $\delta'_{x,y_0} = 1$ and $x^+ \setminus \{y_0\}$ contains neither a projective nor an injective vertex.*

It is well-known that the Auslander–Reiten quiver of an artin algebra together with the usual length function satisfies all the assumptions of the preceeding result. Applying the corollary in this situation one obtains Liu's theorem.

The rest of this paper is devoted to proving our combinatorial result, using a series of six lemmas. We stress that all the assertions of the lemmas can be found in a representation theoretic formulation in Liu's paper [L].

Recall that given a path $y = x_n \to \cdots \to x_1 \to x_0 = x$ of length $n \geq 1$, the vertex y is called a *predecessor* of x and the vertex x is called a *successor* of y. If the path is sectional, i.e. $\tau x_i \neq x_{i+2}$ for all x_i, $0 \leq i \leq n-2$, lying in Γ'_0, then the vertex y is called a *sectional predecessor* of x and the vertex x is called a *sectional successor* of y.

Lemma 1 *Let x be a vertex such that $\tau^r x$ is non-projective for all $r \geq 0$. Then all predecessors of x are non-projective if and only if all sectional predecessors of $\tau^r x$ are non-projective for all $r \geq 0$.*

Proof: Suppose there is a path $x_n \to \cdots \to x_0 = x$ with projective x_n. Choose i minimal such that $y = \tau^s x_i$ is projective for some $s \geq 0$. We obtain a sectional path $y \to \cdots \to \tau^r x$, for some $r \geq s$, by induction on i.

Lemma 2 *Let $x_n \to \cdots \to x_1 \to x_0 = x$ be a sectional path of length $n \geq 1$. If $\sum_{y \in x^-} \delta_{y,x} \ell(y) - \ell(x_1) \geq \ell(x)$, then x_n is non-projective.*