Non-linear combinatorial vectorspace categories

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Introduction

In the study of the representation theory of finite dimensional algebras besides module categories also other sorts of categories arise. These categories are frequently still Krull-Schmidt categories but fail to be abelian. One important class of these categories are the subspace categories of vectorspace categories which appeared classically when modules over one-point extension algebras were considered.

Let us first recall the relevant definitions. We fix an algebraically closed field k. A Krull-Schmidt category K (over k) is a k-additive category such that the morphism space $K(x, y)$ is finite dimensional for all objects x, y of K and the endomorphism algebra $K(x, x)$ is local for each indecomposable object x of K. We fix a set of representatives of the isomorphism classes of indecomposable objects in K and denote the induced full subcategory by $\operatorname{ind} K$.

A vectorspace category (over k) is a pair $\mathcal{K} = (K, |-|)$ consisting of a Krull-Schmidt category K and a k-linear functor $|-| : K \to k\text{-mod}$ where k-mod is the category of finite dimensional k-vectorspaces. Note that we compose morphisms in Krull-Schmidt categories usually from left to right whereas in k-mod we compose from right to left.

The subspace category $\check{\mathcal{U}}(\mathcal{K})$ of a vectorspace category $\mathcal{K} = (K, |-|)$ is again a Krull-Schmidt category whose objects are the triples $U = (U_0, U_\omega, \gamma_U)$ where U_0 is an object of K, U_ω is a finite dimensional k-vectorspace and $\gamma_U : U_\omega \to |U_0|$ is a k-linear map. For two objects U, V of $\check{\mathcal{U}}(\mathcal{K})$ the space of morphism form U to V is the set of pairs $f = (f_0, f_\omega)$ where $f_0 \in K(U_0, V_0)$ and $f_\omega \in \operatorname{Hom}_k(U_\omega, V_\omega)$ satisfying $|f_0|\gamma_U = \gamma_V f_\omega$. The morphisms in $\check{\mathcal{U}}(\mathcal{K})$ are composed componentwise.

The first aim in studying $\check{\mathcal{U}}(\mathcal{K})$ is to understand the indecomposable objects. A part of them can be constructed directly from K. Namely there is a canonical full embedding of K into $\check{\mathcal{U}}(\mathcal{K})$ defined by mapping an object x of K to the triple $(x, 0, 0)$. For general $\mathcal{K} = (K, |-|)$ there may be indecomposable objects x of K such that $|x| = 0$. It is an easy calculation that an indecomposable object U of $\check{\mathcal{U}}(\mathcal{K})$ such that there is an indecomposable direct summand x of U_0 with $|x| = 0$ has to be isomorphic to $(x, 0, 0)$. This shows that the full subcategory of $\check{\mathcal{U}}(\mathcal{K})$ supported by the indecomposables U with this property is equivalent to K. Therefore these objects of $\check{\mathcal{U}}(\mathcal{K})$ are uninteresting and we may suppose without loosing information that $\mathcal{K} = (K, |-|)$ is sincere i.e. $|x| \neq 0$ for all objects x of K.

If one is mainly interested in the indecomposables of a Krull-Schmidt category L one may always pass to a representation equivalent category by factoring out an ideal of L which does not contain an automorphism of x for any object x of L. Applying this to the subspace category $\mathcal{U}(\mathcal{K})$ of a sincere vectorspace category $\mathcal{K} = (K, |-|)$, we may factor out the ideal of all morphisms $f = (f_0, f_\omega)$ such that $|f_0| = 0$. The factor of $\mathcal{U}(\mathcal{K})$ by this ideal coincides with the subspace category of the vectorspace category $(K/\mathrm{Ker}\,|-|, |-|)$. But by construction the functor $|-|$ acts faithfully on the factor category $K/\mathrm{Ker}\,|-|$. So we have seen that for studying the indecomposables of $\mathcal{U}(\mathcal{K})$ we may always suppose that $\mathcal{K} = (K, |-|)$ is not only sincere but even faithful i.e. $|-|$ is a faithful functor.

If A is a one-point extension algebra, the category of finite dimensional A-modules can be identified with the subspace category of some vectorspace category $\mathcal{K} = (K, |-|)$ (see [Ri3]). A big part of research on algebras of finite representation type was based on the analysis of this situation in the case when A is even representation directed (see again [Ri3]). In this situation the occurring vectorspace categories $\mathcal{K} = (K, |-|)$ are always linear i.e. they are faithful and $\dim_k |x| = 1$ holds for each indecomposable object x of K.

The main advantage of linear vectorspace categories $\mathcal{K} = (K, |-|)$ is that one can reconstruct them completely from relatively few data. Namely one defines a partial ordered set $S = S(\mathcal{K})$ by choosing the objects of $\mathrm{ind}\, K$ as elements and putting $x \geq y$ if $K(x, y) \neq 0$. The point is that $\mathcal{K}$ is isomorphic to the vectorspace category $(\mathrm{add}\,(kS), can)$ where $\mathrm{add}\,(kS)$ is the additive closure of the category kS which has as objects the elements of S and between two objects x, y there is a 1-dimensional morphism space with basis element $b(x, y)$ if $x \geq y$ whereas otherwise the morphism space is 0. The composition in this category is defined by linear extending the rule $b(x, y)b(y, z) = b(x, z)$. The canonical functor *can* acts in the obvious way. This means that one can reconstruct a linear vectorspace category completely only from the knowledge when there exist non-zero morphisms between indecomposables.

Dealing with tame type, naturally vectorspace categories which are not linear occur. Nevertheless in certain situations one would like to have similar uniqueness results as in the linear case. Let us be more precise. In [Ri2] and [Ri3] certain tame faithful vectorspace categories $\mathcal{K} = (K, |-|)$ called patterns are considered which have the property that $\dim_k |x| \leq 2$ for all $x \in \mathrm{ind}\, K$. These vectorspace categories are visualized by the quiver of the category $\mathrm{ind}\, K$, which has as points just the objects of $\mathrm{ind}\, K$, and the number $\dim_k |x|$ attached to each of these points in the following way. The possible values for $\dim_k |x|$ are 1 and 2. The points where the value 2 is reached are presented as bold points. By our results in [Dr1] and [Dr2] we were led to vectorspace categories $\mathcal{K} = (K, |-|)$ where we could compute the quiver of $\mathrm{ind}\, K$, some of the imposed relations and the values of $\dim_k |x|$. The images we obtained resembled very much the ones corresponding to patterns. So we asked ourselves whether these data determine the vectorspace categories. Or in other words, is there a model vectorspace category to which all other vectorspace categories with the same data have to be isomorphic? Considering the non-isomorphic vectorspace categories of type $\mathbb{D}_{4\lambda}$ (compare [Ri3]) one can see that such a result cannot be true in complete generality.

In section 1 of this paper we will recall the relevant definitions on Krull-Schmidt categories and their quivers.

In section 2 we will introduce the concept of a pattern-like quiver which is a pair (Γ, T) consisting of a quiver Γ and a subset T of the vertex set of Γ satisfying certain properties. We will define a corresponding category $K_{(\Gamma,T)}$ as factor category of the k-linear path category $k\Gamma$ by an admissible ideal $J_{(\Gamma,T)}$. Furthermore, we will calculate $\dim_k K_{(\Gamma,T)}(x, y)$