INTERMEDIATE SUBGROUPS IN CHEVALLEY GROUPS

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Let G be a group and D be a subgroup of G. In this paper we are interested in the description of the lattice

$$\mathcal{L}(D,G) = \{H \mid D \leq H \leq G\}$$

of subgroups of G which contain D. We call subgroups $H \in \mathcal{L}(D,G)$ subgroups **intermediate** between D and G. We are primarily interested in the parametrization of intermediate subgroups and their behaviour with respect to set-theoretic and group-theoretic operations like intersection, conjugation, passage to normalizer of H or to normal closure of D in H, etc.

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We will speak about the structure of $\mathcal{L}(D,G)$ as opposed to the study of other classes of subgroups, like, for example, subgroups, generated by a given type of elements/subgroups (say, transvections, reflections, root elements, ...), defined by group theoretic properties (say, abelian, nilpotent, soluble, ...), ...

Little can be said about the structure of the lattice $\mathcal{L}(D,G)$ in general. In fact it is proven in [Tu] that *any* algebraic lattice is isomorphic to an interval in the subgroup lattice of an *infinite* group. This is why, to get sensible results, one has to specialize the class of groups considered. It turns out that in amazingly many examples arising in real life the lattice $\mathcal{L}(D,G)$ has a very transparent structure. There are some abstract conditions on a subgroup D, which are responsible for that (see § 10 and 14).

In our talk the group G will be a Chevalley group or its extension by diagonal automorphisms. Many results are only known for the classical groups, and some will be stated only for the general linear group to save space. We concentrate on results which are valid for *arbitrary* fields, not just finite or algebraically closed. This is why we speak only of *normal* types. While for a finite group theorist there is a good reason to consider also the twisted groups, for us they are no better than all other forms of semisimple algebraic groups.

Although we say something about generalizations to *commutative* rings, we do not discuss here results which *essentially* depend on the nature of the ground ring, or use some deeper ring theory. For the linear and unitary groups there are non-commutative generalizations of some of the results in the present paper, which depend on stability conditions, finiteness conditions, or non-commutative localizations. On the other side, for arithmetical rings there is a theory with many results which have no immediate counterparts in the theory we discuss. We leave all these things beyond the horizon. Some indications may be found in [HO], [PR], [Va16], [Va19], [Z2].

Several general remarks are in order here. First, one should note that many of the results we state below are easy – sometimes trivial – for finite fields. However for infinite fields they may be extremely difficult. People working in finite groups do not usually realize how much the finiteness condition – not even the Classification, but such trivial things as counting arguments or Sylow's theorem – make life easier. The most striking example of this is perhaps the theorem of O.King [Ki2] on the overgroups of diagonal subgroup in SL_2.

In some other cases the situation is strictly opposite: the results are easy to prove for infinite fields using Zariski topology and 'general position' arguments – which deliver no information whatsoever for the finite case. Sometimes it is easy to account for both the finite and the infinite case using *different* proofs for finite and infinite cases. However a 'natural' proof