Stable K - theory and topological Hochschild homology of A_∞ rings

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1 Stable K-theory and topological Hochschild homology

The purpose of this note is to expand on material presented by the third-named author in a lecture at Northwestern University in 1988. The material that concerns us is the description of a program to prove that the analog of the trace map induces a natural homotopy equivalence of infinite loop spaces between stable K-theory and topological Hochschild homology of A_∞ ring spectra. B. Dundas and R. McCarthy [2] have given a proof of the result (formulated for simplicial rings), using a strategy totally different from what we will describe here.

In this section we recall the definitions of these objects and prove a few of the basic properties. For us a spectrum will always be a connected strict Ω-spectrum [8, page 11]. The definition of an A_∞-ring spectrum is found in [9, page 248]. Expanding this definition in terms of diagrams satisfying certain properties suggests the definitions of A_∞-module spectra and of A_∞-bimodule spectra over a given A_∞-ring. For details we must refer the reader to [3] and [4]. From now on, a ring is an A_∞-ring spectrum and a module over a ring is an A_∞-module spectrum over the given A_∞-ring, unless we make an explicit qualification. The algebraic K-theory of an A_∞ ring was first defined in [9, page 272]; other definitions of the algebraic K-theory of A_∞-rings are found in [15].

Our definition of stable K-theory is suggested by remarks found in [19, page 388] concerning the stable K-theory of simplicial rings. First comes a piece of notation. If M is a module over R, then $M\langle m\rangle$ is the module M shifted up by m dimensions. More precisely,

$$M\langle m\rangle = M \wedge S^m,$$

where $\wedge$ indicates the smash product of the spectrum with a pointed space [8, page 16].

Definition 1.1 *We let R be a ring and let M be an R-bimodule. Define*

$$K^s(R;M) = \lim_m \Omega^{m+1}\text{fiber}(K(R \oplus M\langle m\rangle) \longrightarrow K(R)).$$

In section 3 we will need to know how well $\Omega^{m+1}\text{fiber}(K(R \oplus M\langle m\rangle) \longrightarrow K(R))$ approximates $K^s(R;M)$. Proposition 1.3 of [22, page 38] generalizes to the A_∞ context and leads to the following result.