Euler systems for higher K-theory of number fields

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1 Introduction

In this paper we construct elements in higher K-groups of a number field which behave similarly to the Euler system of Gauss sums investigated by K.Rubin in [Ru2]. Let p^m be a power of an odd prime number p. Choose a square free integer L whose prime factors split completely in the cyclotomic field $\mathbf{Q}(\mu_{p^m})$. Let $\bar{r}$ be a prime ideal of $\mathbf{Q}(\mu_{p^m L})$ above a rational prime r. For L and $\bar{r}$ as above we construct elements $\Lambda(L, \bar{r})$ which belong to K-groups with coefficients in $\mathbf{Z}/p^m$ of the field $\mathbf{Q}(\mu_L)$. These elements have the following properties which we prove in Section 3 of the paper.

Theorem A

(1) *If* $Tr_l\colon K_{2n+1}(\mathbf{Q}(\mu_L);\mathbf{Z}/p^m) \to K_{2n+1}(\mathbf{Q}(\mu_{L/l});\mathbf{Z}/p^m)$ *is the transfer map in* K*-theory, then:*

$$Tr_l(\Lambda(L,\bar{r})) \;=\; \Lambda(L/l,\hat{r})^{1-l^n Fr_l^{-1}}.$$

Here Fr_l *is the Frobenius automorphism for a prime* l *dividing* L.

(2) *If* $red\colon K_{2n+1}(\mathbf{Q}(\mu_L);\mathbf{Z}/p^{m+1}) \to K_{2n+1}(\mathbf{Q}(\mu_L);\mathbf{Z}/p^m)$ *is the change of coefficients map in* K*-theory, then:*

$$red(\Lambda(L,\bar{r})) = \begin{cases} \Lambda(L,\tilde{r})^p & \text{if } m \geq m_0 \\ \Lambda(L,\tilde{r}) & \text{if } m < m_0, \end{cases}$$

where m_0 *is an integer depending on* r *and* p.

(3) *If* $\partial\colon K_{2n+1}(\mathbf{Q}(\mu_L);\mathbf{Z}/p^m) \to \bigoplus_{\nu|p} K_{2n}(k_\nu;\mathbf{Z}/p^m)$ *is the boundary map in the localization sequence, then:*

$$\partial(\Lambda(L,\bar{r})^{b_L^n\gamma_p}) \;=\; [N_{\bar{r}/\tilde{r}}(\pi_{\bar{r}}(B^n(\xi_{p^m})))]^{\Theta_n(b_L)}.$$

Here $\Theta_n(b_L)$ *denotes the* n*th Stickelberger element defined for an integer* b_L, *and* $N_{\bar{r}/\tilde{r}}(\pi_{\bar{r}}(B^n(\xi_{p^m})))$ *is an element of the group* $K_{2n}(k_{\tilde{r}};\mathbf{Z}/p^m)$.

We would like to mention that our construction of elements $\Lambda(L,\bar{r})$ suggests how to obtain similar elements in K-groups of modular curves. We plan to address the matter elsewhere. Using $\Lambda(L,\bar{r})$ we were able to make a similar reasoning as K.Rubin in [Ru2], to get some results about the size of the group $D_{n+1}(\mathbf{Q})$ of divisible elements in the p-torsion part of $K_{2n}(\mathbf{Q})$ for n odd. A proof of the following statement which we give in Section 4 of this paper does not use "the Main Conjecture" in Iwasawa theory.

Theorem B
The number of elements of $D_{n+1}(\mathbf{Q})$ *divides the* p*-part of the number* $w_{n+1}(\mathbf{Q})\zeta_{\mathbf{Q}}(-n)$ *for primes* p *relatively prime to* $w_{n+1}(\mathbf{Q})$, *where* n *is odd and* $n \not\equiv -1 \mod (p-1)$.

While proving Theorem B [cf. Corollary 3, below] we had to reformulate some of Rubin's results from [Ru2] for the field $\mathbf{Q}(\mu_{p^m})$ and $m > 1$. Our computations show a striking analogy between arithmetic of class groups of cyclotomic fields and arithmetic of the group $D_{n+1}(\mathbf{Q})$. In the last section of the paper we prove some results about the group $D_{n+1}(F)$ for a number field F and n even.

Theorem C
If F *is a totally real, abelian number field and* p *does not divide the conductor of* F, *then we have the following formula for the number of elements in* $D_{n+1}(F)$.

$$|D_{n+1}(F)| \;=\; [K_{2n+1}^{et}(\mathcal{O}_F[\frac{1}{p}])\,;\,C_{2n+1}^{et}]$$

Here C_{2n+1}^{et} *stands for the image in etale* K*-theory of Soulé's group of cyclotomic elements defined for the field* F.