BOUNDS FOR CHARACTERISTIC FUNCTIONS OF POLYNOMIALS IN ASYMPTOTICALLY NORMAL VARIABLES

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Abstract. Let f be a polynomial in k variables and S_n be a normed sum of independent identically distributed random vectors $X_1, X_2, \dots, X_n$ taking values in $\mathbb{R}^k$. Upper bounds for $|\mathbf{E}\exp\{it\, f(S_n)\}|$ are derived provided that either the distribution of X_1 has a nondegenerate discrete component or the distribution of S_{n_0} has an absolutely continuous component for some n_0. The dependence on t and n is given in an explicit form. The proofs are based on some number-theoretic results.

1. Introduction and notation

Let $X = (X^{(1)}, \dots, X^{(k)})$ denote a random vector taking values in a k-dimensional Euclidian space $\mathbb{R}^k$. Consider a sequence $X_1, X_2, \dots$ of independent copies of X. Let

$$S_n = n^{-1/2}(X_1 + \cdots + X_n)$$

and $S_n = (S_n^{(1)}, \cdots, S_n^{(k)})$.
In this paper we study the characteristic function

$$g_n(t, a) = \mathbf{E}\exp\{it\, f(S_n + a)\}, \qquad a \in \mathbb{R}^k,$$

for large values of n and t, where

$$f(x) = \sum_{m_1=0}^{m} \cdots \sum_{m_k=0}^{m} \alpha(m_1, \dots, m_k)\left(x^{(1)}\right)^{m_1} \cdots \left(x^{(k)}\right)^{m_k}, \qquad m \geq 2.$$

Estimates of $g_n(t, a)$ can be based on the so-called symmetrization inequality (see Götze (1979) and Götze (1989), especially Corollary 3.26 and the remark after it referring to Weyl (1916); see also Yurinskii (1981)).

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In the case $k = 1$, for example, this approach leads to the following inequality (cf. Lemmas 3.1, 4.1, 4.2 in Yurinskii (1983)). Let

$$f(x) = \alpha x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_0.$$

For any nondegenerate random variable X, positive numbers A, ε and any integer $m \geq 2$ there exist positive constants c_1 and c_2 depending on A, ε, m and the distribution of X such that for all $a \in \mathbb{R}^1$

$$\big|g_n(t,a)\big| \leq c_1\big(n^{-A} + |\alpha t|^{-1/(m2^m)}\big) \tag{1}$$

provided that

$$c_2|\alpha t| \leq n^{(m-1)/2-\varepsilon}. \tag{2}$$

Remark. If X has a discrete distribution it is well known that $g_n(t,a)$ is an almost periodic function (see e.g. Besicovitch (1932), p.6, and Cramer (1970), p.26) and hence

$$\overline{\lim_{t\to\infty}}\,\big|g_n(t,a)\big| = 1.$$

Thus, $|t|$ has to be restricted to a finite interval depending on n and f.
The aim of this paper is to show that in (1) the exponent $1/(m2^m)$ of $|t|^{-1}$ can be replaced by a larger value in the following two cases:
a) the distribution of X has a nondegenerate discrete component (see Theorems 1 and 2 below),
b) for some n_0 the distribution of S_{n_0} has an absolutely continuous component (see Theorems 3 and 4).
Moreover, we get also bounds for $|\mathbf{E}\exp\{it\,f(Y+a)\}|$, where Y has a Gaussian distribution in $\mathbb{R}^k$ (see Theorems 5, 6 and 7).
Our proofs are based on inequalities for trigonometric sums and integrals.
In the case *a)* we use Vinogradov's type estimates for trigonometric sums (see Prachar (1978) and Lemma 6 below). In the case *b)* we apply the Vinogradov inequality for trigonometric integrals (see Lemma 4, ch.2, section II in Vinogradov (1985)) and its generalizations to the multidimensional case (see Lemma 4 below). Our refinements of (1) are similar in form to the improvements given by the Vinogradov method for trigonometric sums as compared to Weyl's results (see the Introduction of Section II in Vinogradov (1985)).

2. Main results

Let P, P_1, P_2 denote three probability measures in $\mathbb{R}^k$ and let γ be a positive number, $0 < \gamma \leq 1$. The measure P_1 is called a *component* of P of *weight* γ if

$$P = \gamma P_1 + (1-\gamma)P_2.$$

In Theorems 1 and 2 we consider the one-dimensional case and polynomials

$$f(x) = \alpha x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_0, \qquad m \geq 2.$$