χ^*-extremal graphs and the lexicographic product

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Abstract

The star-chromatic number and the fractional-chromatic number are two generalizations of the ordinary chromatic number of a graph. We say a graph G is χ^*-*extremal* if its star-chromatic number is equal to its fractional-chromatic number. We prove that χ^*-extremal graphs G have the following nice property: For an arbitrary graph H the star-chromatic number $\chi^*(G[H])$ of the lexicographic product $G[H]$ is equal to the product of $\chi^*(G)$ and $\chi(H)$. Then we show that several classes of circulant graphs are χ^*-extremal. Thus for these circulant graphs G and arbitrary graphs H, if $\chi^*(G)$ and $\chi(H)$ are known then we can easily determine the star-chromatic number (hence the ordinary chromatic number) of the lexicographic product $G[H]$.

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1 Introduction

Let k and d be positive integers such that $k \geq 2d$. A (k, d)-coloring of a graph $G = (V, E)$ is a mapping $c : V \mapsto Z_k = \{0, 1, \ldots, k-1\}$ such that, for each edge $uv \in E, |c(u) - c(v)|_k \geq d$, where $|x|_k = \min\{|x|, k - |x|\}$. This generalizes the usual notion of a k-coloring: an ordinary k-coloring of G is just a $(k, 1)$-coloring of G. Thus, the chromatic number $\chi(G)$ of G is the smallest k for which there is a $(k, 1)$-coloring. The *star-chromatic number* of G, $\chi^*(G)$, is defined as

$$\chi^*(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-coloring}\}.$$

The star-chromatic number of a graph was first introduced by Vince [10]. His approach uses some methods and results from continuous mathematics. A purely combinatorial treatment of the star-chromatic number of a graph was supplied by Bondy and Hell [1], and an alternate definition of the star-chromatic number can be found in [12]. It was proved in [1, 10], among other things, that

$$\chi(G) - 1 < \chi^*(G) \leq \chi(G), \quad i.e., \quad \chi(G) = \lceil \chi^*(G) \rceil.$$

Thus the chromatic number of a graph is determined by its star-chromatic number. On the other hand, two graphs with the same chromatic number could have different star-chromatic numbers. In this sense, the star-chromatic number of a graph captures its structure more precisely than the ordinary chromatic number.

Another generalization of the ordinary chromatic number is the fractional-chromatic number of a graph.

Definition 1 A mapping c from the collection $\mathcal{S}$ of independent sets of a graph G to the interval $[0, 1]$ is a fractional-coloring if for every vertex x of G we have $\sum_{S \in \mathcal{S} \text{ s.t. } x \in S} c(S) = 1$. The value of a fractional-coloring c is $\sum_{S \in \mathcal{S}} c(S)$. The *fractional-chromatic number* $\chi_f(G)$ of G is the infimum of the values of fractional-colorings of G.