THE HALL AND COMPOSITION ALGEBRAS OF CYCLIC QUIVER

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1. Introduction

Let Q_n be the quiver of type $\widetilde{A}_{n-1}$ of cyclic orientation with n vertices

$$\begin{array}{ccccccc} & & 1 & & 2 & & \\ & & \circ & \rightarrow & \circ & & \\ & \nearrow & & & & \searrow & \\ n\ \circ & & & & & & \circ \\ & \nwarrow & & & & \swarrow & \\ & & \circ & \cdots & \circ & & \end{array}$$

conventionally,indices concerning with the vertices of Q_n are taken from $\mathbf{Z}/n\mathbf{Z}$, the residue calsses of integers modulo n.

Let k be a field and $\mathcal{T}$ be the category of locally nilpotent finite-dimensional representations of Q_n. As is remarked in [12], $\mathcal{T}$ may be considered as the category of finite length modules over some ring, and we will call the objects in $\mathcal{T}$ modules. Let S_i be the simple module corresponding to the vertex i of Q_n . Let M be any module and $\mathbf{r}M$ be the radical of M, let $\mathbf{r}^{t+1}M = \mathbf{r}(\mathbf{r}^tM)$, if M is indecomposable with top S_i, then it has an unique filtration of submodules

$$M \supset \mathbf{r}M \supset \cdots \supset \mathbf{r}^tM \supset \mathbf{r}^{t+1}M \cdots \supset \mathbf{r}^{j-1}M \supset \mathbf{r}^jM = 0$$

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such that $\mathbf{r}^t M/\mathbf{r}^{t+1}M \cong S_{i+t}$ for $t = 0, 1, \cdots, m-1$ and M is unique up to isomorphism with this property since any indecomposable module in $\mathcal{T}$ is serial. Let $M_{i,j}$ be the unique indecomposable R module with top S_i and composition length j, and $\{[M_{i,j}] | 1 \le i \le n, 1 \le j \le \infty\}$ is a complete set of the isomorphism classes of indecomposable R modules. By the Krull-Schmidt theorem,each M in $\mathcal{T}$ can be written uniquely as a direct sum of indecomposable representations

$$M \cong \oplus_{i=1}^n \oplus_{j=1}^m a_{i,j} M_{i,j}$$

where $a_{i,j}$ is a nonnegative integer and $a_{i,j}M_{i,j}$ denotes a direct sum of $a_{i,j}$ copies of $M_{i,j}$,and $a_{i,j}$ is unique determined by M. Let $\mathbf{Z}^+$ be the set of nonnegative integers,and let $((\mathbf{Z})^+)^{n\infty} = \{\mathbf{a} = (a_{i,j}) | 1 \le i \le n, 1 \le j \le \infty, a_{i,j} \in \mathbf{Z}^+, a_{i,j} \ne 0$ for all but finite many $j\}$. We have a bijection from the isomorphism classes of modules in $\mathcal{T}$ onto $((\mathbf{Z})^+)^{n\infty}$,

$$M = \oplus_{i=1}^n \oplus_{j=1}^m a_{i,j} M_{i,j} \longmapsto \mathbf{a}(M) = (a_{i,j})$$

with the inverse

$$\mathbf{a} = (a_{i,j}) \longmapsto M(\mathbf{a}) = \oplus_{i=1}^n \oplus_{j=1}^m a_{i,j} M_{i,j}.$$

The following order is defined in [1]. Assume that $\mathbf{a} = (a_{i,j}), \mathbf{b} = (b_{i,j}) \in ((\mathbf{Z})^+)^{nm}$.

Define $\mathbf{a} \prec_2 \mathbf{b}$ if $\sum_{i=1}^n \sum_{j=1}^m j a_{i,j} < \sum_{i=1}^n \sum_{j=1}^m j b_{i,j}$, or $\sum_{i=1}^n \sum_{j=1}^m j a_{i,j} = \sum_{i=1}^n \sum_{j=1}^m j b_{i,j}$ and there is a pair (s, t) such that $a_{i,j} = b_{i,j}$ for all $j > t$ or $j = t$ and $i < s$ and $a_{s,t} > b_{s,t}$.

Since there are only finite many $\mathbf{a} = (a_{i,j}) \in ((\mathbf{Z})^+)^{n\infty}$ such that $\sum_{i=1}^n \sum_{j=1}^m j a_{i,j} = d$ for a given nonnegative integer d, $\prec_2$ is a total order of the semi-additive group $((\mathbf{Z})^+)^{nm}$ with the smallest element $(0, 0, \cdots, 0)$. And $((\mathbf{Z})^+)^{n\infty}$ is totally ordered semi-additive group with respect to these orders.

In the rest of this paper, we assume that k is a field of q elements.