Semigroups containing proximal linear maps

by

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I. Introduction

A linear automorphism of a finite dimensional real vector space is called *proximal* if it has a unique eigenvalue – counting multiplicities – of maximal modulus. Proximal elements play an important role in the study of the dynamics of linear maps because of their simple structure: The induced map on projective space has a unique attracting fixed point whose basin consists of the whole space except for a hyperplane. This contraction property played an essential role in Tits' proof of what is now called the Tits alternative [Ti], in Furstenberg's work on boundary theory [F], in the paper of the two last named authors on maximal subgroups [MS], in the work of Guivar'ch and coauthors on random walks on groups (cf. [GR]). The authors were motivated to do this work by their hope for yet another application of proximality, namely in their work in progress on the Auslander conjecture. For work on this conjecture see [A, Mi, D, DG, FG, GK, GrM, Ma1-3, Me, To, S].

An efficient tool to find proximal elements is the following corollary of the main algebraic result of [GM]: Given a subgroup G of $GL(V)$ which contains a proximal element. Then so does every Zariski dense subsemigroup H of G, provided V – considered as a G–module – is strongly irreducible (for a definition see the paragraph preceding 3.12). Here we show that under these hypotheses H in fact provides for a rich supply of proximal elements: There is a finite subset M of H such that for every $g \in GL(V)$ at least one of the elements γg, $\gamma \in M$, is proximal. We obtain such a subset M with at most $(\dim V)^2$ elements. We show the main result (theorem 4.1) for a quantitative version of proximality, called (r, ε)–proximality. For another richness result see 4.10.

In section 5 of the paper we extend our result in two directions, namely for direct sums of strongly irreducible representations and for other types of Lyapunov filtrations. The latter means that instead of requiring that G contains an element with exactly one eigenvalue of maximal modulus we look at the minimum number n_1 of eigenvalues of maximal modulus among elements of G . Then the same minimum is attained in the Zariski dense subsemigroup H . Similarly for the eigenvalues with maximum and next to maximum modulus, etc.

In the final section we give necessary and sufficient conditions when the image of a reductive group under an irreducible representation ρ contains a proximal linear map. The conditions are phrased in terms of the highest weight of ρ .

As an application (actually of 4.1 and results of [GR]) we obtain (see 6.8) that for the group $G(\mathbb{R})$ of real points of a connected semi–simple algebraic group G over $\mathbb{R}$ every Zariski dense subsemigroup H contains $\mathbb{R}$ –regular elements, where an element $g \in G(\mathbb{R})$ is called $\mathbb{R}$ –regular if the number of eigenvalues – counting multiplicities – of modulus 1 of $\mathrm{Ad}(g)$ is minimum possible among elements $g \in G(\mathbb{R})$. This was known before for sub*groups* H ([BL], [P]). We also obtain a corresponding finiteness result.

The proofs of the finiteness results 4.1 and 5.14 make essential use of the notion of quasiprojective transformation due to Furstenberg [F] and the methods of the algebraic part of [GM]. The relevant notions and results are given in section 3. The proof of 4.1 is contained in section 4. There the simple structure of proximal linear maps is expounded, technically speaking in the form of "contractions" and "contractive sequences" on projective space. In section 5 we give the generalization to several irreducible summands and more general Lyapunov filtrations.

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