A SPECIAL CLASS OF 2-GENERATOR KLEINIAN GROUPS

Elena Klimenko

Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld, Deutschland

and

Department of Mathematics
Novosibirsk State University
Novosibirsk 630090, Russia

Abstract. In this article we continue investigation of a special class of 2-generator Kleinian groups in which neither of the generators nor their commutator is strictly loxodromic. We give necessary and sufficient conditions for discreteness of non-elementary and non-Fuchsian groups of this class generated by two hyperbolic elements.

Introduction

Let $\Gamma = \langle A, B\rangle$ be a two-generator subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We study the following question: when Γ is discrete?

The same question for two-generator subgroups of $\mathrm{PSL}(2,\mathbb{R})$ has a problematic history in the mathematical literature. There were a lot of papers dealing with this problem, see [Br2], [DJ], [Gi1] – [Gi3], [GiM], [J], [KIR], [Kn], [LU], [Mt], [P1] – [P3], [PR], [R1] – [R5]. Rosenberger [R5] completed the classification of all two-generator Fuchsian groups and presented an algebraic algorithm for determining whether or not a two-generator subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is discrete. Another, geometric approach to this problem was started by Matelski [Mt] and developed by Gilman and Maskit [GiM].

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As for $\mathrm{PSL}(2,\mathbb{C})$, the problem is far from being completely solved. The remarkable Jørgensen's inequality [J] yields a necessary condition for discreteness of 2-generator non-elementary groups.

Necessary and sufficient discreteness conditions for subgroups of $\mathrm{PSL}(2,\mathbb{C})$ which are generated by two elliptic elements with mutually orthogonal skew axes were given in [K1], [K2]. Thus, some class of 2-generator Kleinian groups was first completely described. Maskit [M2] investigated another specific class of 2-generator Kleinian groups, where one of generators has exactly two fixed points in $\overline{\mathbb{C}}$ and the other maps one of these fixed points to the other. There were two open questions in [M2]. The answer to one of them is given in [K3]. There are discrete groups belonging to both classes studied in [K2] and [M2], however, neither of these classes contains the other.

We study a larger class $\mathcal{G}$ of 2-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$, where neither of the generators nor their commutator is strictly loxodromic. This class contains all 2-generator Fuchsian groups as well as all Kleinian groups described in [K2] and [M2]. There are a lot of non-elementary discrete groups which are not Fuchsian in $\mathcal{G}$. Most of such groups have been detailed in [K4], see also [K5]. Moreover in [K4] a description of these groups in terms of generators and relations is given, and all orbifolds that are uniformized by the groups are also presented.

Some examples of discrete two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ have been constructed in [HMV].

In this paper we give necessary and sufficient conditions for discreteness of non-elementary and non-Fuchsian groups of class $\mathcal{G}$ generated by two hyperbolic elements (Theorem 4).

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$\mathrm{PSL}(2,\mathbb{C})$ and Lobachevsky space

As is known, $\mathrm{PSL}(2,\mathbb{C})$ acts as the full group of orientation preserving isometries of 3-dimensional Lobachevsky space $\mathbb{H}^3$.

Consider 3-dimensional Lobachevsky space realized as the upper half-space

$$\mathbb{H}^3 = \{\ (z,t) \mid z \in \mathbb{C},\ t > 0\}$$

with the hyperbolic metric

$$ds^2 = (|dz|^2 + dt^2)/t^2.$$