$THH(R) \cong R \otimes S^1$ for E_∞ Ring Spectra

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1 Introduction and Main Results

Around 1985 M. Bökstedt introduced the notion of Topological Hochschild Homology THH of a functor with smash products [2]. The category of such functors is topologically enriched. In a conversation Bökstedt pointed out that if this category were tensored $THH(R)$ ought to be the tensor $R \otimes S^1$ in the commutative case.
Recall that a category is *topologically enriched* if the morphism sets are topologized such that composition is continuous. In a topologically enriched category we have the notions of indexed limits and colimits.

1.1 Definition: Let $\mathcal{K}$ and $\mathcal{B}$ be topologically enriched categories and let $F : \mathcal{K} \to \mathcal{T}op$, $G : \mathcal{K}^{op} \to \mathcal{T}op$ and $X : \mathcal{K} \to \mathcal{B}$ be continuous functors. The *limit of X indexed by F* is an object $\lim_F X$ in $\mathcal{B}$ together with a natural homeomorphism

$$\mathcal{B}(B, \lim_F X) \cong Funct(\mathcal{K}, \mathcal{T}op)(F, \mathcal{B}(B, X(-)))$$

where $Funct(\mathcal{K}, \mathcal{T}op)$ denotes the category of continuous functors $\mathcal{K} \to \mathcal{T}op$. The *colimit of X indexed by G* is an object $\operatorname{colim}_G X$ in $\mathcal{B}$ together with a natural homeomorphism

$$\mathcal{B}(\operatorname{colim}_G X, B) \cong Funct(\mathcal{K}^{op}, \mathcal{T}op)(G, \mathcal{B}(X(-), B))$$

If F and G are the constant functors to a point we get the usual definitions of limits and colimits with the additional requirement that the natural bijections be homeomorphisms. If $\mathcal{K}$ consists of one object and its identity and F and G take the space K as value while X takes $B \in ob\mathcal{B}$ as value, we denote $\lim_F X$ by B^K and $\operatorname{colim}_G X$ by $B \otimes K$.

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1.2 Definition: If $B \otimes K$ exist for all $B \in \mathcal{B}$ and $K \in \mathcal{T}op$, B is called *tensored.* If all B^K exist it is called *cotensored.*

In view of Bökstedt's remark and its implications tensored and cotensored structures are our central concern. They have the following universal property

1.3 Let $\mathcal{B}$ be a topologically enriched tensored and cotensored category. Then for $B_1, B_2 \in \mathcal{B}$ and $K \in \mathcal{T}op$ we have natural homeomorphisms

$$\mathcal{B}(B_1 \otimes K, B_2) \cong \mathcal{T}op(K, \mathcal{B}(B_1, B_2)) \cong \mathcal{B}(B_1, B_2^K)$$

This shows that $B \otimes K$ is the K-indexed sum of copies B if K is discrete and B^K the K-indexed product. Hence $B \otimes K$ and B^K are topologically parametrized versions of K-indexed sums and products.
¿From (1.3) we immediately deduce

1.4 For $B \in \mathcal{B}$ and $K, L \in \mathcal{T}op$ we have natural isomorphisms

$$\begin{array}{lcl} (B \otimes K) \otimes L & \cong & B \otimes (K \times L) \\ (B^K)^L & \cong & B^{K \times L} \end{array}$$

The central idea of Bökstedt's definition of THH is to take the classical definition of Hochschild homology, replace ring by a suitable notion of ring spectrum and the tensor product by the smash product. At that time there was no known category of spectra with an associative, commutative and unital smash product. So Bökstedt introduced unstably defined functors with smash products whose structures allow a stabilization procedure, and a fairly small coherence machinery took care of a coherently homotopy associative, commutative and unital smash product of such functors.
Recently Elmendorf, Kriz and May [6] discovered a category of spectra which admits a strictly (up to natural isomorphisms) associative, commutative and unital smash product making it into a symmetric monoidal category. Moreover, it has the pleasant property that the A_∞ ring spectra and E_∞ ring spectra, which are structured by the linear isometry operad $\mathcal{L}$, are exactly the monoids respectively commutative monoids in this category. Hence it is very simple to define THH in terms of this smash product $\star$ (see §4), and therefore we will work in this setting. It has not been established yet that this definition of THH agrees with the one of Bökstedt, but this is very likely to be true [15], at least for cofibrant ring spectra in the sense of [6, §5].