Volumes of the Fibonacci manifolds

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Abstract

This paper is devoted to the study of the compact hyperbolic 3-manifolds uniformized by the Fibonacci groups. It is shown that their volumes are equal to volumes of non-compact hyperbolic 3-manifolds arising as complements of some well-known knots. All these volumes are described in terms of the Lobachevsky function.

1. Introduction.

We consider the three dimensional compact orienatable hypebolic manifolds connected with the Fibonacci groups. The Fibonacci groups

$$F(2, m) = < x_1, x_2, ..., x_m : x_i x_{i+1} = x_{i+2}, \; i \bmod m >$$

were introduced by J. Conway [2]. First of all the following question was connected with these groups: whether $F(2, m)$ are infinite groups or not. From [3], [6], [10], [8] and [19] it is known that the group $F(2, m)$ is finite if and only if $m = 1, 2, 3, 4, 5, 7$. The algebraic generalizations of $F(2, m)$ were considered in [5].

A new stage in the investigation of the Fibonacci groups is connected with a remakable work of Helling, Kim and Mennicke [8]. They showed that the group $F(2, 2n)$, $n \geq 4$, is isomorphic to a discontinuous co-compact subgroup of

$PSL_2(\mathbb{C})$, the full group of orientation-preserving isometries of the hyperbolic space $\mathbb{H}^3$. Moreover the group $F(2,6)$ is isomorphic to a 3-dimentional affine group.

We will call hyperbolic manifolds $M_n = \mathbb{H}^3/F(2,2n)$, $n \geq 4$, uniformized by Fibonacci groups to be **Fibonacci manifolds**.

In [9] it is shown that manifolds M_n are n–fold cyclic covers of the S^3, branched over the figure-eight knot. We note that M_n are isometric to hyperbolic manifolds, discribed by Hempel [7].

We continue the investigation of the algebraical, topological and arithmetical properties of Fibonacci manifolds. We consider the properties of the volumes of the Fibonacci manifolds.

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2. Hyperbolic volumes. Thurston–Jørgensen theorem.

In this section we recall some properties of the volumes of the hyperbolic manifolds. An n-dimensional hyperbolic manifold can be defined as a quotient space $M^n = \mathbb{H}^n/\Gamma$, where Γ is a discrete isometry group of the Lobachevsky space $\mathbb{H}^n$ acting without fixed points. The concepts of hyperbolic area and hyperbolic volume in $\mathbb{H}^2$ and $\mathbb{H}^3$ are carried over naturally to M^2 and M^3. We are interested in the set $\mathcal{M}^n$, $n = 2,\ 3$, of n-dimensional orientable hyperbolic manifolds of the finite volume.

Consider the volume function $v_n : \mathcal{M}^n \to \mathbb{R}$, $n = 2, 3$ which makes correspondence between a manifold $M^n \in \mathcal{M}^n$ and its volume $vol(M^n)$. It is interesting to remark that the volume functions v_2 and v_3 have essentially different properties.

The two-dimensional case is described by the Gauss–Bonnet theorem. If M^2 is a hyperbolic surface of genus g with k points removed, then

$$vol(M^2) = 2\pi\,(2g - 2 + k).$$

Therefore the values of v_2 are positive numbers from the set $2\pi\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, and we have the following picture for the values of v_2 (see Figure 1):