Tutte's 5-flow conjecture for graphs of nonorientable genus 5

Eckhard Steffen *

Abstract

We develop four constructions for nowhere-zero 5-flows of 3-regular graphs which satisfy special structural conditions. Using these constructions we show a minimal counter-example to Tutte's 5-flow conjecture is of order ≥ 44 and therefore every bridgeless graph of nonorientable genus ≤ 5 has a nowhere-zero 5-flow.

One of the structural properties is formulated in terms of the structure of the multigraph $G(\mathcal{F})$ obtained from a given 3-regular graph G by contracting the circles of a 2-factor $\mathcal{F}$ in G.

1 Introduction

All terminology and notation stated not explicitely follows that of Bondy and Murty [1]. Let G be a graph, $\phi : EG \to \{0, 1, \cdots, k-1\}$ be a function and D be an *orientation* of G, i.e. an assignment of a direction to each edge. We define for each vertex $v \in VG$,

$$\phi(v) = \sum_{uv \in D} \phi(uv) + \sum_{vu \in D} -\phi(vu) \tag{1}$$

to be the *weight of v under ϕ* and call (D, ϕ) a *k-weighting* of G. The *deficit set* $\Delta_{(D,\phi)}$ of (D, ϕ) is the set of vertices of G, where $\phi(v) \neq 0$. If $\Delta_{(D,\phi)} = \emptyset$, the k-weighting (D, ϕ) is called a *k-flow* of G. If, in addition, $\phi(e) \neq 0$ for each $e \in EG$ then (D, ϕ) is a *nowhere-zero k-flow*. In 1954 Tutte [6] conjectured the following,

5-Flow Conjecture: *Every bridgeless graph has a nowhere-zero 5-flow.*

There are several results which approximate this conjecture. The best one is due to Seymour [4]. He proved

Theorem 1.1 *Every bridgeless graph has a nowhere-zero 6-flow.*

*Fakultät für Mathematik, SFB 343, Postfach 100131, 33501 Bielefeld, Germany

The next theorem relates edge 3-colorings to nowhere-zero 4-flows for 3-regular graphs, see e.g. [2]. It shows the 5-flow conjecture is best possible.

Theorem 1.2 *A 3-regular graph G has a nowhere-zero 4-flow iff it is edge 3-colorable.*

A bridgeless 3-regular graph, which is not edge 3-colorable is a *snark*, e.g. the Petersen graph is a snark. To give an indirect proof of the 5-flow conjecture, the structure of a possible minimal counter-example has been the subject of many investigations. Celmins proved the following result, see e.g. [2].

Theorem 1.3 *Every minimal counter-example to the 5-flow conjecture is a cyclically 5-edge-connected snark of girth at least 7.*

Steinberg [5] showed the 5-flow conjecture is true for the projective plane. Möller, Carstens and Brinkmann [3] improved this result by proving

Theorem 1.4 *Every 3-regular graph of orientable genus ≤ 2 or non-orientable genus ≤ 4 has a nowhere-zero 5-flow.*

They checked all 3-regular graphs of order ≤ 28 and with girth ≥ 7 with a computer and applied the following lemma.

Lemma 1.1 *Let G be a 3-regular graph embedded in a surface S of minimal genus γ, such that there is no face having a boundary of length < 7. Then the following holds:*
(a) If S is orientable then $|VG| \leq 28(\gamma - 1)$.
(b) If S is non-orientable then $|VG| \leq 14(\gamma - 2)$.

However the lower bound of the order of a minimal counter-example can be easily deduced from the girth requirement and Theorem 1.5, which was proved by Jaeger [2]. It is a consequence of his more general considerations on "nearly F_4-graphs".

Theorem 1.5 *Every bridgeless 3-regular graph with a 2-factor having 0 or 2 odd components has a nowhere-zero 5-flow.*

Since the basic problem is difficult, one direction of research is to prove special results, see Jaeger [2], Section 10. In this paper we pursue this approach and extend the result of Möller, Carstens and Brinkmann [3]. We show

Theorem 1.6 *Every 3-regular graph of orientable genus ≤ 2 or non-orientable genus ≤ 5 has a nowhere-zero 5-flow.*