A Note on Comparison Theorems of Splittings and Multisplittings of Hermitian Positive Definite Matrices

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Abstract. We discuss iterative methods for the solution of the linear system $Ax = b$, which are based on a single splitting or a multisplitting of A. In order to compare different methods, it is common to compare the spectral radius of the iterative matrix. For M–matrices A and weak regular splittings there exist well-known comparison theorems. Here, we give a comparison theorem for splittings of Hermitian positive definite matrices. Furthermore, we establish a comparison theorem for multisplittings of a Hermitian positive definite matrix.

§1 Introduction and Definitions

If we consider linear systems of equations $Ax = b$, $A \in \mathbb{C}^{n,n}, x, b \in \mathbb{C}^n$, that occur in the numerical analysis, in many cases the matrix $A = [a_{ij}]$ is an M–matrix, i.e. $A \in \mathbb{R}^{\mathrm{n,n}}$, $\mathrm{a_{ij}} \leq 0$ for $i \neq j$, A^{-1} exists and is nonnegative, or A is a Hermitian positive definite matrix. Stimulated by O. Taussky in 1958 [T], a lot of common properties of these two classes of matrices were found, as for instance, determinant inequalities for principal submatrices of A. Here, we consider iterative methods for approximating the solution of the linear system $Ax = b$. We establish results for Hermitian positive definite matrices, which corresponds with well-known results for M–matrices.

One class of iterative methods is based on a *splitting* (M, N) of A, i.e. $A = M - N$, M nonsingular. Then, the iteration has the form

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b \quad i = 0, 1, 2, \ldots \tag{1.1}$$

It is well-known that (1.1) converges for all x_0 if and only if $\rho(M^{-1}N) < 1$ [V], [BP]. Here, $\rho(M^{-1}N)$ denotes the spectral radius of $M^{-1}N$.

A splitting (M, N) of A is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, and *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$. Furthermore, Ortega [O] called a splitting *P-regular* if $M^H + N$ is Hermitian positive definite. Here, we use the notation $M \geq 0$ for a matrix to be nonnegative and $M \succ 0$ ($M \succeq 0$) for a matrix to be Hermitian positive (semi-) definite. For both sets of splittings, there exist well-known convergence theorems for the sequence (1.1):

Theorem 1.1 (Varga [V]) *Let $A \in \mathbb{R}^{n,n}$ and (M, N) be a weak regular or a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.*

Theorem 1.2 (Householder - John [H],[J]) *Let $A \in \mathbb{C}^{n,n}$ be Hermitian and let (M, N) be a P-regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A \succ 0$.*

In order to get a parallel algorithm for the sequence (1.1), O'Leary and White introduced in [OW] the concept of *multisplittings*. They called $(M_k, N_k, E_k)_{k=1}^m$ a multisplitting of A if for each k (M_k, N_k) is a single splitting of A, and E_k is a nonnegative diagonal matrix with $\sum_{k=1}^m E_k = I$. Here, I denotes the identity matrix. Then, the iteration to solve $Ax = b$ becomes

$$x^{i+1} = \sum_{k=1}^{m} E_k(M_k^{-1}N_k x^i + M_k^{-1}b). \tag{1.2}$$

Using the matrices

$$H = \sum_{k=1}^{m} E_k M_k^{-1} N_k, \quad G = \sum_{k=1}^{m} E_k M_k^{-1}, \tag{1.3}$$

we can express, if G is nonsingular, the multisplitting $(M_k, N_k, E_k)_{k=1}^m$ as a single splitting $(G^{-1}, G^{-1}H)$ of A. Multisplittings have been discussed in several recent papers, for example the papers of Neumann and Plemmons [NP], White [W1], [W2], Elsner [E] and Elsner, Bru and Neumann [BEN]. O'Leary and White established in [OW] the following theorem:

Theorem 1.3 *i) If, for $k = 1, \ldots, m$, (M_k, N_k) is a weak regular splitting of a matrix A satisfying $A^{-1} \geq 0$, then (1.2) is convergent.*
ii) If, for $k = 1, \ldots, m$ (M_k, N_k) is a P-regular splitting of a symmetric positive definite matrix A and $E_k = \alpha_k I$, then (1.2) is convergent.
iii) If, for $k = 1, \ldots, m$, $||(M_k, N_k)||_\infty < 1$, then (1.2) is convergent. Here, $|| \cdot ||_\infty$ is the maximum row sum matrix norm.

Moreover, White gives in [W2] some more convergence results for Hermitian positive definite matrices satisfying a special zero pattern, i.e. matrices in dissection form or in nested dissection form. But here, we consider general Hermitian positive definite matrices and general multisplittings.

In order to compare the asymptotic convergence rate of (1.1) and (1.2), it is common to compare $\rho(M^{-1}N)$ for different single splittings or $\rho(H)$ for different multisplittings. For M–matrices, or more general for monotone matrices, and weak regular splittings there exist well-known comparison theorems. A matrix A is called *monotone*, if A^{-1} exists and $A^{-1} \geq 0$. In this note, we give a comparison theorem for single and multisplittings of Hermitian positive