Optimal Detection of a Counterfeit Coin with Multi–Arms Balances*

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Abstract

We consider the problem of locating a light coin out of a set containing n coins, $n-1$ of which have the same weight. The weighing device is a balance with $r \geq 2$ pans that, when r equally sized subset of coins are weighted, indicates the subset eventually containing the light coin. We give an algorithm to find the counterfeit coin that requires the minimum possible average number of weighings. All previous results on this problem considered two-arms balances only.

1 Introduction

The problem of locating a counterfeit coin out of a set of n coins, $n-1$ of which are good, is one of the oldest search problems ever studied and it is often used in introductory texts on the design and analysis of algorithms as a "paradigmatic example" (cfr. [15]). Many papers have been written on the counterfeit coin problem and several weighing models have been considered. For an account of the vast literature on the subject we refer the interested reader to [1], [2], [4], [10] and references therein quoted. Problems of searching for more than one coin have been studied in [3], [5], [6], [8], [11], [14], [16].

In one of the most popular models one is given a two–arms balance scale with which to compare the weights of two equally sized subsets of coins. The balance will tell us whether the subsets have the same weight, or the first set is lighter (and therefore contains the light coin), or

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the second one is lighter. The problem is to locate the counterfeit coin using as few weighings as possible. Two measures are commonly utilized to estimate the goodness of an algorithm: The worst–case number of weighings and the average number of weighings needed to locate the counterfeit coin; in the latter case it is assumed that one is given a probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$, where p_i is the probability that the i–th coin is counterfeit[1]. Moreover, two classes of algorithms are usually considered: *Sequential* (or *adaptive*) algorithms and *predetermined* (or *non–adaptive*) algorithms. In sequential algorithms the weighing performed on the i–th step depends upon the feedbacks (outcomes) of the previous $i-1$ weighings, while in predetermined algorithms all the weighings are fixed beforehand (for more on these questions see [1]).

Sequential and predetermined optimal algorithms that locate a counterfeit coin using the minimum *worst–case* number of weighings are presented in ([1], Ch. 2). Optimal sequential algorithms requiring the *minimum average* number of weighings are given in [13] and [17], whereas Linial and Tarsi [12] find average-case optimal algorithms, both for the sequential and predetermined cases, for a variant of the classical model. More precisely, in [12] it is assumed that the counterfeit coin can be either lighter or heavier and this is not known *a priori*.

All above papers considered two–arms balances only. As an interesting generalization, Aigner [1] has proposed the problem of considering an r–arms balance scale ($r \geq 2$) such that when r equally sized subsets of coins are weighted in parallel, it indicates which subsets, if any, contains the lighter coin. Karp *et al.* [9] investigated the closely related problem of locating a given number k of defectives, but they considered a much powerful device that is able to weigh in parallel *any* r subsets of coins (not necessarily equally sized); the outcome indicates which of the subsets contain at least one defective coin. In his book Aigner [1] presents an optimal sequential algorithm requiring minimum worst–case number of weighings for the r–arms balance scale and states as an open problem that of finding an optimal sequential algorithm requiring minimum average number of weighings. In this paper we solve this problem thus also generalizing the result of [17] from $r = 2$ to arbitrary r.

We also show that the uniform distribution is the "worst" possible probability distribution on the set of coin, in the sense that for any other probability distribution the minimum average number of weighings is upper bounded by the minimum average number of weighings necessary when the uniform probability distribution is assumed.

[1] In this paper we assume, as done also in [12], [13] and [17], that the probability distribution $\mathbf{p}$ is the uniform one; however see also Section 3.