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ON THE UPPER BOUND OF THE SIZE OF THE r-COVER-FREE FAMILIES

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ABSTRACT

Let $T(r,n)$ denote the maximum number of subsets of an n-set satisfying the condition in the title. It is proved in a purely combinatorial way, that for n sufficiently large

$$\frac{\log_2 T(r,n)}{n} \leq 8 \cdot \frac{\log_2 r}{r^2}$$

holds.

1. Introduction

The notion of the r-cover-free families was introduced by Kautz and Singleton in 1964 [17]. They initiated investigating binary codes with the property that the disjunction of any $\leq r$ ($r \geq 2$) codewords are distinct (UD_r codes). This led them to studying the binary codes with the property that none of the codewords is covered by the disjunction of $\leq r$ others (*Superimposed codes, ZFD_r codes;* P. Erdős, P. Frankl and Z. Füredi called the correspondig set system r-cover-free in [7]).

Since that many results have been proved about the maximum size of these codes. Various authors studied these problems basically from three different points of view, and these three lines of investigations were almost independent of each other. This is why many results were found first in information theory ([1], [4], [5], [14], [15], [16], [17]), were later rediscovered in combinatorics ([2], [6], [7], [10], [18], [19]), or in group testing ([12], [13]), and vice versa.

We shall approach this area from the combinatorial side. Our main goal is to estimate the maximal size of the family of subsets of an n-element set with the property that no set is covered by the union of r others.

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2. Notation, definitions

Let S be an n-element set. 2^S is the set of all subsets of S. $\binom{S}{k}$ denotes the set of all k-subsets of S ($k \geq 0$). If $|S| = n$, then $|\binom{S}{k}| = \binom{n}{k}$. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$, and $\log x$ is always of base 2. A set system $\mathcal{A} \subseteq 2^S$ is called k-uniform if its members are k- sets. It is usually supposed that the underlying set of the set systems is $[n]$.

We call $\mathcal{F}' \subset 2^S$ r-distinct, if

$$\bigcup_{i=1}^{k} A_i \neq \bigcup_{j=1}^{\ell} B_i$$

for any

$$\{A_1, A_2, \ldots, A_k\} \neq \{B_1, B_2, \ldots, B_\ell\},$$

$1 \leq k, \ell \leq r$; $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_\ell \in \mathcal{F}'$. $\mathcal{F} \subset 2^S$ is r-cover-free, if

$$A_0 \not\subseteq A_1 \cup A_2 \cup \ldots \cup A_r$$

holds for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$. $\mathcal{F}^\star \subset 2^S$ is $< r$ part intersecting, if

$$|A_i \cap A_j| < \frac{1}{r} \min\{|A_i|, |A_j|\}$$

for any distinct $A_i, A_j \in \mathcal{F}^\star$ holds. We denote by $T'(r,n)$, $T(r,n)$, $T^\star(r,n)$ and $T'(r,n,k)$, $T(r,n,k)$, $T^\star(r,n,k)$ the maximum cardinality of the corresponding set systems in general and in k-uniform case, resp. We will provide upper bounds on these functions for r fixed and n tending to infinity. We shall call

$$R(r) = \limsup_{n \to \infty} \frac{\log T(r,n)}{n}$$

the *rate* of the r-cover-free family. The following proposition is obvious from the definitions.

Proposition 2.1. *If $\mathcal{F}$ is $< r$ part intersecting, then $\mathcal{F}$ is r-cover-free; and if $\mathcal{F}$ is r-cover-free, then $\mathcal{F}$ is r-distinct. Hence*

$$T^\star(r,n) \leq T(r,n) \leq T'(r,n), \qquad \text{and} \qquad T^\star(r,n,k) \leq T(r,n,k) \leq T'(r,n,k).$$

The following upper and lower bounds were proved in [1], [4], [5], [7], [13]: there exist two (absolute) constants c_1, c_2 such that

$$\frac{c_1}{r^2} \leq \frac{\log T(r,n)}{n} \leq \frac{c_2}{r} \tag{1}$$

for any n. In most papers the lower bound is proved by probabilistic methods. In [13] V.T. Sós and F.K. Hwang used a greedy-type algorithm to generate $< r$ part intersecting