On the exclusion of forest minors: a short proof of the path-width theorem

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Robertson and Seymour proved that excluding any fixed forest F as a minor imposes a bound on the path-width of a graph. We give a short proof of this, reobtaining the best possible bound of $|F|-2$.

1. Introduction

At the start of their Graph Minors series, Robertson and Seymour [1] proved, by a long and involved argument, that for every forest F there exists an integer n such that every graph without an F minor has path-width at most n. This bound was brought down to the best possible value by Bienstock, Robertson, Seymour and Thomas [2], who proved the following:

Theorem 1. [2] *For every forest F, every graph of path-width $\geqslant |F|-1$ has a minor isomorphic to F.*

The authors remark that this result is best possible in two ways. First, the value of $|F|-1$ is sharp, because the complete graph K_{n-1} has path-width $n-2$ but no (forest) minor on n vertices. Second, if F is *not* a forest then the exclusion of F as a minor does not bound the path-width of a graph: as noted without proof in [1], trees can have arbitrarily large path-width (but will never contain F as a minor if F contains a cycle).

The proof of Theorem 1 in [2], already much shorter than [1], relies on a non-trivial minimax theorem involving the concept of "blockages". (These were adapted from "tangles", a central concept in the Graph Minors series concerning tree-width.) Although interesting in its own right, it turns out that this minimax theorem is not needed for a proof of Theorem 1; the purpose of this note is to give a short direct proof.

2. Definitions

Graphs in this paper are finite, and they may have loops or multiple edges. Let G be a graph. If $X \subset V(G)$ or $X \subset G$, then $G[X]$ denotes the subgraph of G induced by the vertices in X. Following [2], we denote by $\mathrm{att}(X)$ (for "attachment") the set of those vertices in X that have a neighbour in $G - X$, and write $\alpha(X) := |\mathrm{att}(X)|$. A *minor* of G is a graph obtained from a subgraph of G by contracting edges.

A *path-decomposition* of G is a sequence $(W_1, \ldots, W_s)$ of subsets of $V(G)$ such that

(i) $W_1 \cup \ldots \cup W_s = V(G)$, and for every edge e of G there exists an $r \leqslant s$ such that both endvertices of e are in W_r;

(ii) $W_p \cap W_r \subset W_q$ whenever $1 \leqslant p \leqslant q \leqslant r \leqslant s$.

The *width* of a path-decomposition as above is the number

$$\max\{\, |W_r| - 1 : 1 \leqslant r \leqslant s \,\},$$

and the *path-width* of G is the smallest width of any path-decomposition of G.

For each positive integer n, we denote by $\mathcal{B}_n = \mathcal{B}_n(G)$ the unique minimal subset of the power set of G satisfying the following two conditions:

(i) $\emptyset \in \mathcal{B}_n$;

(ii) if $X \in \mathcal{B}_n$, $X \subset Y \subset V(G)$ and $\alpha(X) + |Y \backslash X| \leqslant n$, then $Y \in \mathcal{B}_n$.

Thus, a set $X \subset V(G)$ is in $\mathcal{B}_n$ if and only if there is a sequence

$$\emptyset = X_0 \subset \ldots \subset X_s = X$$

such that $\alpha(X_r) + |X_{r+1} \backslash X_r| \leqslant n$ for all $r < s$.

For example, if $(W_1, \ldots, W_s)$ is a path-decomposition of G of width $< n$, then all the sets $W_1 \cup \ldots \cup W_r$ for $r \leqslant s$, including $V(G)$ (for $r = s$), are in $\mathcal{B}_n$. This is easy to verify by induction on r from the axioms of a path-decomposition, but will not be used below. Its converse is also true, and will be needed later:

(2.1) *If $V(G) \in \mathcal{B}_n$ then G has path-width $< n$.*

Indeed, if $V(G) \in \mathcal{B}_n$ then there is a sequence $\emptyset = X_0 \subset \ldots \subset X_s = V(G)$ such that $\alpha(X_r) + |X_{r+1} \backslash X_r| \leqslant n$ for all $r < s$. With

$$W_r := \mathrm{att}(X_{r-1}) \cup (X_r \backslash X_{r-1})$$

for all $1 \leqslant r \leqslant s$, the sequence $(W_1, \ldots, W_s)$ is easily seen to be a path-decomposition of G of width $< n$.