The accuracy of Gaussian approximations in nilpotent Lie groups

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We consider the Central Limit Theorem with Gaussian limit distributions in stratified nilpotent Lie groups. We obtain estimates of the rate of convergence and Edgeworth expansions for expectations of smooth functionals.

1. Introduction

The role of nilpotent Lie groups in investigations of the Central Limit Theorem (CLT) in compact topological groups may be explained by an important result due to Hazod and Siebert [24], [8]. They showed that if a probability measure on a locally compact topological group has a nonempty domain of attraction then its support lies in a subgroup isomorphic to a simply connected nilpotent Lie group whose Lie algebra admits a positive graduation.

In this paper we find refinements of the following version of the CLT due to Raugi [19] (see also the survey paper Pap [16]). Let $|\cdot|$ denote a homogeneous norm on G, let $\delta_t \in \mathrm{Aut}(G)$, $t > 0$ denote a semigroup of standard dilations, and let μ be a centered probability measure on G. For $t > 0$ define the measure $\delta_{1/t}\mu(A) = \mu(\delta_t A)$, where $A \subset G$. Denote by μ^n the n-fold convolution of μ. Then the condition $\int |x|^2 \mu(dx) < \infty$ implies that the measures $\mu_n = \delta_{1/\sqrt{n}}\mu^n$, $n = 1, 2, 3, \ldots$ converge weakly to a Gaussian measure, say ν.

Under the second moment assumption in Section 4 a Berry-Esseen type bound is given for $\int f(x)(\mu_n - \nu)(dx)$ provided $f : G \to \mathbb{R}$ is a sufficiently smooth function. In particularly, this bound implies Raugi's version of the CLT.

In Section 5 we obtain the Edgeworth expansion

$$\int f(x)(\mu_n - \nu)(dx) = \alpha n^{-1/2} + O(n^{-1}),$$

where the coefficient $\alpha = \alpha(G, f, \mu)$ depends on G, several lower order moments of μ and derivatives of f only.

Sections 2 and 3 contain some preliminaries about Lie groups, Gaussian measures and differential calculus on these groups (see the book of Heyer [11] for the related information). The stratified Taylor expansion plays an important role. Notice that the proofs of the results

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rely heavily on differential calculus, although they are formally independent of a concrete version of it. Therefore we choose and adopt a version of this calculus most convenient for our purposes.

Our main aim was to develop methods and technique applicable in the non-commutative case. The approach here is related to the Lindeberg method as it is described in Bentkus et al. [1]. Due to the non-commutativity a direct application of this method is not possible. In this paper we restrict ourselves to the class of stratified nilpotent Lie groups of step s. This class seems to be sufficiently rich to reflect the structure of general nilpotent Lie groups. We restrict ourselves as well to investigation of $\int f(x)(\mu_n - \nu)(dx)$ with smooth f only since already in this case algebraic diffuculties are present in the full extent. As usually the investigation of the Berry-Esseen bounds and Edgeworth expansions for probabilities requires additionally ingenious analytic technique, and we are going to publish related results elsewhere. Remark that the only result known on nilpotent Lie groups is a Berry-Esseen type bound for probabilities of homogeneous balls due to Pap [14]. This bound is obtained under relatively stringent moment assumptions whereas the assumptions in this paper seem to be optimal. Edgeworth expansions on Lie groups are considered here for the first time. Remark that the term α in the Edgeworth expansion has certain new features specific only for the non-commutative case. The corresponding classical results for the group $(\mathbb{R}^k, +)$ follow from our theorems substituting $s = 1$.

The rates of convergence due to Khokhlov [12] and Rachev and Yukich [18] are not comparable with our ones since they are obtained for the Euclidean motion group, which is the semidirect product of $(\mathbb{R}^k, +)$ and of the compact group of orthogonal transformations of $\mathbb{R}^k$.

2. Preliminaries on nilpotent Lie groups

The letters c, C (respectively $c(\cdot)$, $C(\cdot)$) with or without indices will denote positive generic constants (respectively positive constants depending only on the quantities in parentheses); the same symbol may stand for different constants.

Let $\mathcal{G}$ be a m-dimensional s-step nilpotent Lie algebra with descending central series

$$\mathcal{G}^{(1)} = \mathcal{G}, \quad \mathcal{G}^{(2)} = [\mathcal{G}, \mathcal{G}^{(1)}], \ \ldots, \ \mathcal{G}^{(k+1)} = [\mathcal{G}, \mathcal{G}^{(k)}], \ \ldots, \ \mathcal{G}^{(s+1)} = \{0\}.$$

Let V_k be the vector subspace complementary to $\mathcal{G}^{(k+1)}$ in $\mathcal{G}^{(k)}$. Then $\mathcal{G} = \oplus_{k=1}^{s} V_k$ is a vector space decomposition. $\mathcal{G}$ is called *stratified* if V_1 generates $\mathcal{G}$ as an algebra. Let $\{X_1, \ldots, X_m\}$ be a basis in $\mathcal{G}$ adapted to the above decomposition, i. e. there exists a mapping

$$d : \{1, 2, \ldots m\} \to \{1, 2, \ldots, s\}$$