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On a Search Problem in Multidimensional Grids

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ABSTRACT

A search problem of G. O. H. Katona [2] was solved in [1] where the unknown element $\mathbf{x}$ *is a point of a 2-dimensional lattice, and the queries are of type "is* $\mathbf{x} = (x_1, x_2) \leq \mathbf{a} = (a_1, a_2)$*?", where* $\mathbf{a}$ *is an arbitrary lattice point and* $\mathbf{x} \leq \mathbf{a}$ *means that* $a_i \leq b_i$ $(i = 1, 2)$*. In the recent paper we consider the generalization of this problem for arbitrary dimension* d*.*

1. Introduction

The following model of the combinatorial search theory was initiated by A. Rényi. A finite set X and a family $\mathcal{A}$ of its subsets are given. We need to find an unknown element x in X by asking questions of type "is x in A?", for any member A of $\mathcal{A}$. Our goal is to minimize the number of questions to identify x in the worst or in the average case.

There are two different approaches of the described model. In the first we have to ask all queries in advence and from the answers we have to identify x. This is called *parallel search.* In the second case the queries are asked sequentially, all questions may depend on the previous answers. This is called an *adaptive search.* In the recent paper we shall give a worst case analysis in the adaptive model for a special search problem.

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Consider the well known fact, that in the adaptive case it is sufficient and necessary to ask $\lceil \log n \rceil$ questions of type "is $x \le b$?" for some $b \in X$ to find an unknown element x in an n-element ordered set X. But what can we say if X is partially ordered rather than ordered? G. O. H. Katona [2] asked this problem for partial orders that are direct products of chains i. e. when X is the set of bounded integer vectors of a fixed dimension and we must find an unknown vector $\mathbf{x} \in X$ by asking queries of the form "is $\mathbf{x} \le \mathbf{b}$?". (See next section for formal definition.)

In [1] M. Ruszinkó considered this problem in dimension two and solved it in most cases. In this paper we consider the same problem in arbitrary dimension d. So now we want to find an unknown integer vector $\mathbf{x} = (x_1, \ldots, x_d)$ bounded by $1 \le x_i \le a_i$ for $i = 1, \ldots, d$. The queries allowed are of the form "is $\mathbf{x} \le \mathbf{b}$?" for an integer vector $\mathbf{b} = (b_1, \ldots, b_d)$ where answer "yes" means $x_i \le b_i$ for all $i = 1, \ldots, d$. It is easy to see that it is necessary to ask $\lceil \sum_{i=1}^{d} \log a_i \rceil$ questions and it is sufficient to ask $\sum_{i=1}^{d} \lceil \log a_i \rceil$ queries (see Proposition 2.1). Thus the gap between these two trivial bounds is at most $d-1$. But which bound is closer to the correct value?

Chapter 2 contains the notations, definitions and trivial cases. In Chapter 3 we improve on the trivial upper bound and prove that in "most" lattices the lower bound is tight. On the other hand we show, that for a certain sequence of lattices $\Omega(d)$ more queries are needed than the trivial lower bound. In Chapter 4 we prove a theorem for the special 2-dimensional case generalizing Theorem 3.3 of [1] giving lots of examples where the lower bound is not tight. In Chapter 5 some open problems are posed.

2. Notations, Definitions, Trivial Cases

For $c \in \mathbf{R}$ we denote by $\lceil c \rceil$ the smallest integer $\ge c$, and by $\lfloor c \rfloor$ the largest integer $\le c$, $\{c\} = c - \lfloor c \rfloor$. We denote by log the logarithm of base 2.

Let $d \ge 1$ be a fixed integer and let us consider d-dimensional integer vectors $\mathbf{a} = (a_1, \ldots, a_d)$. We say that $\mathbf{a} \le \mathbf{b}$ if $a_i \le b_i$ for every $i = 1, \ldots, d$. Let $\mathbf{1} = (1, \ldots, 1)$. We consider the following search problem for a vector $\mathbf{a} \ge \mathbf{1}$.

Problem. An unknown integer vector $\mathbf{1} \le \mathbf{x} \le \mathbf{a}$ has to be found by asking questions of the form "is $\mathbf{x} \le \mathbf{b}$?" for an arbitrary integer vector $\mathbf{b}$. The questions can be asked adaptively, i. e. later questions can depend on the answers for earlier questions. What is the minimum number of questions needed to find the unknown vector $\mathbf{x}$ in the worst case?

Let us denote this minimum by $f(\mathbf{a})$, i. e. it is the minimum for all search strategies of the maximum for all vectors $\mathbf{1} \le \mathbf{x} \le \mathbf{a}$ of the number of queries used to find $\mathbf{x}$.

We call the bounds on the function f given by the following proposition the *trivial lower and upper bounds.*