MacLane Homology and Topological Hochschild Homology

Z. Fiedorowicz, T. Pirashvili, R. Schwänzl, R. Vogt, F. Waldhausen

1 Introduction

In 1985 Bökstedt [2] introduced the topological Hochschild homology functor THH which is related to Waldhausen's algebraic K-theory of certain A_∞ rings [25] in a way similar to the relationship of Hochschild homology of rings to Quillen K-theory: there is a generalization of the trace map. If R is a genuine ring, the trace map factors

$$\begin{tikzcd} K_*(R) \arrow[r] & K_*^s(R) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & HH_*(R) \\ & & THH_*(R) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

through Bökstedt's map β. $K_*^s(R)$ is Waldhausen's stable K-theory of R [26].

It was a surprise when Pirashivili brought MacLane homology H_*^{ML} into the picture, a homology theory for rings developed 30 years earlier by Mac Lane [15]. He observed [19] that α factors as

$$\begin{tikzcd} & H_*^{ML}(R) \arrow[dr] & \\ K_*^s(R) \arrow[ur, "\delta"] \arrow[rr, "\alpha"] & & HH_*(R) \end{tikzcd}$$

and in all calculated examples δ turned out to be an isomorphism. This motivated him to conjecture that MacLane homology is stable K-theory. Waldhausen changed this conjecture to "MacLane homology is topological Hochschild homology" based on the "brave new algebra" argument we will explain in Section 3 below. The two conjectures were related at the time by the long standing conjecture that β is an isomorphism. (This last conjecture has recently been proved for simplicial rings [3], see also [22].) Subsequently Pirashvili and Waldhausen could show [20]

(1.1) Theorem: For rings R there is a natural isomorphism $THH_*(R) \cong H_*^{ML}(R)$

In fact, they proved a more general result allowing coefficients in a functor from the category $\mathcal{P}(R)$ of finitely generated projective R-modules to the category of all R-modules.

The result is established indirectly by showing that $THH_*(R)$ satisfies a set of axioms characterizing the homology of $\mathcal{P}(R)$ with coefficients in the Hom-functor and the fact, due to

Jibladze and Pirashvili [14], that the same is true for $H^{ML}(R)$. This situation is somewhat unsatifactory because it does not really explain the relationship of the two functors.

The purpose of this paper is threefold. Firstly we want to construct an explicit map between MacLane homology and topological Hochschild homology. Secondly we want to analyze the intriguing Q-construction of Eilenberg and Mac Lane used in the definition of MacLane homology, an analysis necessary for the construction of the comparison map but also of separate interest. Thirdly we want to introduce the reader to arguments in "brave new algebra" which in our case really illuminate why MacLane homology and topological Hochschild homology ought to be the same.

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2 Definitions and the Main Result

(2.1) Hochschild homology: For a ring R and an R-bimodule M let $B_*(R,M)$ denote the simplicial abelian group $B_*(R,M) = M \otimes R^{\otimes n}$ with boundary maps

$$\begin{array}{lll} d^i(m \otimes r_1 \otimes \ldots \otimes r_n) & = mr_1 \otimes r_2 \otimes \ldots \otimes r_n & i = 0 \\ & = m \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n & 0 < i < n \\ & = r_n m \otimes r_1 \otimes \ldots \otimes r_{n-1} & i = n \end{array}$$

and degeneracies

$$s_i(m \otimes r_1 \otimes \ldots \otimes r_n) = (m \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_n).$$

Hochschild homology $HH_*(R,M)$ is defined to be the homology of the normalized chain complex $N(B_*(R,M))$ of the simplicial group $B_*(R,M)$ (e.g. see [17, Cor. 22.3]).

To extend this definition to differential graded algebras A we replace R by A, consider M as a trivial complex concentrated in 0, replace $\otimes$ by the tensor product of complexes and arrive at a simplicial chain complex. Regard this as a chain complex of complexes with differential $\partial = \sum\limits_{k=0}^{n_i} (-1)^k \delta^k$.

Hochschild homology of a chain algebra A with coefficients in an A-module M, $HH_*(A,M)$ is the homology of the total complex of this bicomplex (the homology of the condensation in the sense of [16]).

(2.2) MacLane homology: Let C_n denote the set of vertices of the n-dimensional unit cube $I^n = [0,1]^n$. For an abelian group G let $G[C_n]$ denote the set of all maps $g : C_n \to G$. Think of g as the n cube with an element of G at each vertex. Define maps

$$U_i, L_i, S_i : G[C_n] \to G[C_{n-1}]$$

$i = 1, \ldots, n$ as follows: U_i is the restriction to the ith upper face, L_i the restriction to the ith lower face, while S_i adds the ith upper face vertexwise to the ith lower face, i.e.

$$\begin{aligned} U_i(g)(\varepsilon_1, \ldots, \varepsilon_{n-1}) &= g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon_i, \ldots, \varepsilon_{n-1}) \\ L_i(g)(\varepsilon_1, \ldots, \varepsilon_{n-1}) &= g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_i, \ldots, \varepsilon_{n-1}) \\ S_i(g)(\varepsilon_1, \ldots, \varepsilon_{n-1}) &= g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon_i, \ldots, \varepsilon_{n-1}) + g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_i, \ldots, \varepsilon_{n-1}) \end{aligned}$$