Representation Varieties of the Fundamental Groups of Compact Orientable Surfaces

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Abstract
We show that the representation variety for the surface group in characteristic zero is (absolutely) irreducible and rational over **Q**.

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Introduction

Let Γ be a finitely generated group. For any algebraic group G the set $\mathbf{R}(\Gamma, G)$ of all representations (= homomorphisms) $\rho : \Gamma \to G$ is known to have a natural structure of an algebraic variety, and endowed with this structure is called *the variety representations of* Γ *in* G (cf. [Lu-M], [Pl-R]). In the case $G = \mathbf{GL}_n$ which is analyzed by the classical representation theory, $\mathbf{R}(\Gamma, \mathbf{GL}_n)$ is denoted simply by $\mathbf{R}_n(\Gamma)$ and called *the variety of* n*-dimensional representations* of Γ. Since $\mathbf{R}_n(\Gamma)$ is defined by the equations arising from the relations for the generators of Γ, a special role in this theory is played by the one-relator groups

$$\Gamma =< x_1, \ldots, x_n \mid r = 1 > .$$

The methods of this paper allow to consider in full the case : $n \geq 4$, $r = r_1[x_{n-3}, x_{n-2}]$ $[x_{n-1}, x_n]$ where $[x, y] = xyx^{-1}y^{-1}$ is the commutator of x and y, and r_1 is an arbitrary word in the derived subgroup of the free group $F(x_1, \ldots, x_{n-4})$. The most notable groups of this kind are the fundamental groups Γ_g of compact orientable surfaces of genus $g > 1$, and that is why we formulate our results (which remain valid also for $g = 1$) for these groups. So, let $\Gamma_g (g \geq 1)$ be the group with $2g$ generators $x_1, y_1, \ldots, x_g, y_g$ and a single defining relation

$$[x_1, y_1] \ldots [x_g, y_g] = 1.$$

Then a description of $\mathbf{R}_n(\Gamma_g)$ for the ground field of characteristic 0 is given by

Theorem 1 $\mathbf{R}_n(\Gamma_g)$ *is an (absolutely) irreducible* $\mathbf{Q}$*-rational variety of dimension*

$$\dim \mathbf{R}_n(\Gamma_g) = \begin{cases} (2g-1)n^2 + 1 & \textit{if}\, g > 1, \\ n^2 + n & \textit{if}\, g = 1. \end{cases}$$

Informally speaking, Theorem 1 means that "almost all" n-dimensional representations of Γ_g can be parametrized by some rational functions thus yielding a nice description of the totality of representations of Γ_g. However to complete in a sense the representation theory for Γ_g one should supplement the latter with a description of the equivalence classes of representations. In geometric terms this amounts to the analysis of the corresponding variety $\mathbf{X}_n(\Gamma_g)$ of n*-dimensional characters.* Recall that $\mathbf{X}_n(\Gamma)$ can be defined as a categorical quotient of $\mathbf{R}_n(\Gamma)$ modulo the action of $\mathbf{GL}_n$ by conjugation and that the points of $\mathbf{X}_n(\Gamma)$ are in one-to-one correspondence with the equivalence classes of fully reducible representations of Γ (cf.[Lu-M]). (Another realization of $\mathbf{X}_n(\Gamma)$ is given in [Pl].)

Theorem 2 *The character variety* $\mathbf{X}_n(\Gamma_g)$ *is irreducible and* $\mathbf{Q}$*-unirational, of dimension* $(2g-2)n^2 + 2$ *(resp.,* $2n$ *) for* $g > 1$ *(resp.,* $g = 1$*). Moreover,* $\mathbf{X}_n(\Gamma_g)$ *is* $\mathbf{Q}$*-rational if* $g > 1$ *and* $n \leq 3$.