Symmetric chain decompositions of linear lattices

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Introduction

Symmetric chain decompositions were introduced by de Bruin, Tengbergen, and Kruyswijk [1] as a tool for an alternative proof of Sperner's Theorem on the maximal size of an antichain in a finite power set [7]. Symmetric chain decompositions turned out to be useful for several problems, therefore the existence and construction of symmetric chain decompositions in different partially ordered sets has been a standing question in combinatorics (e. g. see [3]). Griggs proved in [6] that the lattice of subspaces of a finite vector space has a symmetric chain decompostion. Since his proof was indirect, Greene and Kleitman formulated the problem of giving an explicit construction for a symmetric chain decomposition of these linear lattices [5]. A partial solution (with restrictions for the underlying field of the vector space) was given by Fleischer [4], a student of Leeb. In this paper we introduce an inductive construction for a symmetric chain decompostion of the lattice of subspaces of a finite dimensional vector space over an arbitrary field. The construction is, in a certain sense, an extension of the well known construction for boolean lattices [5].

Notation

Let us recall first that a *symmetric chain* of a ranked partially ordered set P of finite length n is a chain $x_k \prec x_{k+1} \prec \ldots \prec x_{n-k}$, where the rank of each x_i is i. Then a *symmetric chain decomposition* of P is a partition of P into symmetric chains.

For the following, let $\mathcal{F}$ be a field. All matrices are matrices with entries in $\mathcal{F}$. Then E_t denotes the $(t \times t)$–unit–matrix, whereas 0 denotes the zero–matrix, which, by definition, is a matrix whose entries are just zeros; by abuse of language the matrix 0 always has the right dimensions.

An $(n \times k)$–matrix A with rows $a_1, \ldots, a_n$ is in *Schubert–normal–form* (or short: is an $(n \times k)$–Schubert–normal–form) if

(1) there are $i_1 < i_2 < \ldots < i_k$ such that a_{i_j} equals the j-th k–unit–vector e_j, and

(2) each a_i is a linear combination of the $e_1, \ldots, e_l$ with $i_l \leq i$.

$$
\begin{pmatrix}
0 & \multicolumn{4}{c}{\cdots\cdots\cdots} & 0 \\
\vdots & & & & & \vdots \\
0 & \multicolumn{4}{c}{\cdots\cdots\cdots} & 0 \\
1 & 0 & \multicolumn{3}{c}{\cdots\cdots} & 0 \\
* & 0 & \multicolumn{3}{c}{\cdots\cdots} & 0 \\
\vdots & \vdots & & & & \vdots \\
* & 0 & \multicolumn{3}{c}{\cdots\cdots} & 0 \\
0 & 1 & 0 & \multicolumn{2}{c}{\cdots} & 0 \\
* & * & 0 & \multicolumn{2}{c}{\cdots} & 0 \\
\vdots & \vdots & \vdots & & & \vdots \\
* & * & 0 & \multicolumn{2}{c}{\cdots} & 0 \\
0 & 0 & 1 & 0 & .. & 0 \\
 & & & & \ddots & \\
* & \multicolumn{3}{c}{\cdots\cdots} & * & 0 \\
0 & \multicolumn{3}{c}{\cdots\cdots} & 0 & 1 \\
* & \multicolumn{4}{c}{\cdots\cdots\cdots} & * \\
\vdots & & & & & \vdots \\
* & \multicolumn{4}{c}{\cdots\cdots\cdots} & *
\end{pmatrix}
$$

Figure 1: A matrix in Schubert–normal–form

Observe that the columns of each $(n \times k)$–Schubert–normal–form A generate a k–dimensional subspace U of $\mathcal{F}^n$, the n–dimensional vector space over $\mathcal{F}$. More precisely, A describes the embedding of $\mathcal{F}^k$ into $\mathcal{F}^n$ which maps $\mathcal{F}^k$ onto U. It is easy to see that for each k–dimensional subspace U of $\mathcal{F}^n$ there is exactly one Schubert–normal–form whose colums generate U. Hence there is a bijection between the subspaces of $\mathcal{F}^n$ and the Schubert–normal–forms over $\mathcal{F}$ with n rows. The inclusion of subspaces of $\mathcal{F}^n$ can be described as follows: Let A be an $(n \times k)$– and B be an $(n \times m)$–Schubert–normal–form. Then the subspace generated by A is a subspace of the subspace generated by B if and only if there is an $(m \times k)$–Schubert–normal–form C with $A = B \cdot C$. For more information on Schubert–normal–forms, see [2].

$\mathcal{L}\binom{n}{k}$, where $k \leq n$, denotes the set of $(n \times k)$–Schubert–normal–forms, i. e., this is the set of k–dimensional subspaces of $\mathcal{F}^n$. Then $\mathcal{L}(n) = \bigcup_{k=0}^{n} \mathcal{L}\binom{n}{k}$ is the *n–dimensional linear lattice* over $\mathcal{F}$, i. e., the lattice of subspaces of $\mathcal{F}^n$. The lattice structure of $\mathcal{L}(n)$ can be described just by matrix–multiplication as explained above.

In the sequel, vectors $a \in \mathcal{F}^n$ are viewed as *column–vectors*, a' denotes the transposed vector (tranposed matrix, resp.), i. e., a' is a row–vector with n columns.

The crucial definition is the definition of the sets $\mathcal{L}_t\binom{n}{k}$ and $\mathcal{L}_t(n)$. These are defined as follows:

Definition 1 Let $t \geq 0$ and let

$$
\begin{aligned}
\mathcal{L}_t\binom{n}{k} &= \left\{ \left(\begin{array}{c|c} E_t & 0 \\ \hline A & B \end{array} \right) \in \mathcal{L}\binom{t+n}{t+k} \;\middle|\; \operatorname{rank}(A) = t \right\} , \\
\mathcal{L}_t(n) &= \bigcup_{k=0}^{n} \mathcal{L}_t\binom{n}{k} .
\end{aligned}
$$